CHAPTER 5

Counting

5.1. Basic Principles

5.1.1. The Rule of Sum. If a task can be performed in \( m \) ways, while another task can be performed in \( n \) ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in \( m + n \) ways.

Set theoretical version of the rule of sum: If \( A \) and \( B \) are disjoint sets \((A \cap B = \emptyset)\) then
\[
|A \cup B| = |A| + |B|.
\]
More generally, if the sets \( A_1, A_2, \ldots, A_n \) are pairwise disjoint, then:
\[
|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|.
\]

For instance, if a class has 30 male students and 25 female students, then the class has \( 30 + 25 = 45 \) students.

5.1.2. The Rule of Product. If a task can be performed in \( m \) ways and another independent task can be performed in \( n \) ways, then the combination of both tasks can be performed in \( mn \) ways.

Set theoretical version of the rule of product: Let \( A \times B \) be the Cartesian product of sets \( A \) and \( B \). Then:
\[
|A \times B| = |A| \cdot |B|.
\]
More generally:
\[
|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdots |A_n|.
\]

For instance, assume that a license plate contains two letters followed by three digits. How many different license plates can be printed? Answer: each letter can be printed in 26 ways, and each digit can be printed in 10 ways, so \( 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 676000 \) different plates can be printed.
Exercise: Given a set $A$ with $m$ elements and a set $B$ with $n$ elements, find the number of functions from $A$ to $B$.

5.1.3. The Inclusion-Exclusion Principle. The inclusion-exclusion principle generalizes the rule of sum to non-disjoint sets.

In general, for arbitrary (but finite) sets $A$, $B$:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$  

Example: Assume that in a university with 1000 students, 200 students are taking a course in mathematics, 300 are taking a course in physics, and 50 students are taking both. How many students are taking at least one of those courses?

Answer: If $U = \text{total set of students in the university}$, $M = \text{set of students taking Mathematics}$, $P = \text{set of students taking Physics}$, then:

$$|M \cup P| = |M| + |P| - |M \cap P| = 300 + 200 - 50 = 450$$

students are taking Mathematics or Physics.

For three sets the following formula applies:

$$|A \cup B \cup C| =$$

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|,$$

and for an arbitrary union of sets:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = s_1 - s_2 + s_3 - s_4 + \cdots \pm s_n,$$

where $s_k = \text{sum of the cardinalities of all possible } k\text{-fold intersections of the given sets}$.
5.2. Combinatorics

5.2.1. Permutations. Assume that we have \( n \) objects. Any arrangement of any \( k \) of these objects in a given order is called a permutation of size \( k \). If \( k = n \) then we call it just a permutation of the \( n \) objects. For instance, the permutations of the letters \( a, b, c \) are the following: \( abc, acb, bac, bca, cab, cba \). The permutations of size 2 of the letters \( a, b, c, d \) are: \( ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, dc \).

Note that the order is important. Given two permutations, they are considered equal if they have the same elements arranged in the same order.

We find the number \( P(n, k) \) of permutations of size \( k \) of \( n \) given objects in the following way: The first object in an arrangement can be chosen in \( n \) ways, the second one in \( n - 1 \) ways, the third one in \( n - 2 \) ways, and so on, hence:

\[
P(n, k) = n \times (n - 1) \times \cdots \times (n - k + 1) = \frac{n!}{(n-k)!},
\]

where \( n! = 1 \times 2 \times 3 \times \cdots \times n \) is called “\( n \) factorial”.

The number \( P(n, k) \) of permutations of \( n \) objects is

\[
P(n, n) = n!.
\]

By convention \( 0! = 1 \).

For instance, there are \( 3! = 6 \) permutations of the 3 letters \( a, b, c \). The number of permutations of size 2 of the 4 letters \( a, b, c, d \) is \( P(4, 2) = 4 \times 3 = 12 \).

*Exercise:* Given a set \( A \) with \( m \) elements and a set \( B \) with \( n \) elements, find the number of one-to-one functions from \( A \) to \( B \).

5.2.2. Combinations. Assume that we have a set \( A \) with \( n \) objects. Any subset of \( A \) of size \( r \) is called a combination of \( n \) elements taken \( r \) at a time. For instance, the combinations of the letters \( a, b, c, d, e \) taken 3 at a time are: \( abc, abd, abe, acd, ace, ade, bcd, bce, bde, cde \), where two combinations are considered identical if they have the same elements regardless of their order.
The number of subsets of size $r$ in a set $A$ with $n$ elements is:

$$C(n, r) = \frac{n!}{r!(n - r)!}.$$  

The symbol \(^n\text{C}_r\) (read “$n$ choose $r$”) is often used instead of $C(n, r)$.

One way to derive the formula for $C(n, r)$ is the following. Let $A$ be a set with $n$ objects. In order to generate all possible permutations of size $r$ of the elements of $A$ we 1) take all possible subsets of size $r$ in the set $A$, and 2) permute the $k$ elements in each subset in all possible ways. Task 1) can be performed in $C(n, r)$ ways, and task 2) can be performed in $P(r, r)$ ways. By the product rule we have $P(n, r) = C(n, r) \times P(r, r)$, hence

$$C(n, r) = \frac{P(n, r)}{P(r, r)} = \frac{n!}{r!(n - r)!}.$$