5.3. Generalized Permutations and Combinations

5.3.1. Permutations with Repeated Elements. Assume that we have an alphabet with k letters and we want to write all possible words containing n_1 times the first letter of the alphabet, n_2 times the second letter,..., n_k times the kth letter. How many words can we write? We call this number $P(n; n_1, n_2, \dots, n_k)$, where $n = n_1 + n_2 + \cdots + n_k$.

Example: With 3 a's and 2 b's we can write the following 5-letter words: *aaabb*, *aabab*, *abaab*, *baaab*, *aabba*, *ababa*, *baaba*, *abbaa*, *babaa*, *bbaaa*.

We may solve this problem in the following way, as illustrated with the example above. Let us distinguish the different copies of a letter with subscripts: $a_1a_2a_3b_1b_2$. Next, generate each permutation of this five elements by choosing 1) the position of each kind of letter, then 2) the subscripts to place on the 3 a's, then 3) these subscripts to place on the 2 b's. Task 1) can be performed in $P(5; 3, 2)$ ways, task 2) can be performed in 3! ways, task 3) can be performed in 2!. By the product rule we have $5! = P(5; 3, 2) \times 3! \times 2!$, hence $P(5; 3, 2) = 5!/3!\,2!$.

In general the formula is:

$$P(n; n_1, n_2, \dots, n_k) = \frac{n!}{n_1!\, n_2! \, \dots \, n_k!}\,.$$

5.3.2. Combinations with Repetition. Assume that we have a set A with n elements. Any selection of r objects from A, where each object can be selected more than once, is called a *combination of n objects taken r at a time with repetition.* For instance, the combinations of the letters a, b, c, d taken 3 at a time with repetition are: *aaa*, *aab*, *aac*, *aad*, *abb*, *abc*, *abd*, *acc*, *acd*, *add*, *bbb*, *bbc*, *bbd*, *bcc*, *bcd*, *bdd*, *ccc*, *ccd*, *cdd*, *ddd*. Two combinations with repetition are considered identical if they have the same elements repeated the same number of times, regardless of their order.

Note that the following are equivalent:

1. The number of combinations of n objects taken r at a time with repetition.

2. The number of ways r identical objects can be distributed among n distinct containers.

3. The number of nonnegative integer solutions of the equation:

$$x_1 + x_2 + \cdots + x_n = r\,.$$

Example: Assume that we have 3 different (empty) milk containers and 7 quarts of milk that we can measure with a one quart measuring cup. In how many ways can we distribute the milk among the three containers? We solve the problem in the following way. Let x_1, x_2, x_3 be the quarts of milk to put in containers number 1, 2 and 3 respectively. The number of possible distributions of milk equals the number of non negative integer solutions for the equation $x_1 + x_2 + x_3 = 7$. Instead of using numbers for writing the solutions, we will use strokes, so for instance we represent the solution $x_1 = 2, x_2 = 1, x_3 = 4$, or $2 + 1 + 4$, like this: $||+|+||||$. Now, each possible solution is an arrangement of 7 strokes and 2 plus signs, so the number of arrangements is $P(9; 7, 2) = 9!/7!\,2! = \binom{9}{7}$.

The general solution is:

$$P(n+r-1; r, n-1) = \frac{(n+r-1)!}{r!\,(n-1)!} = \binom{n+r-1}{r}\,.$$

5.4. Binomial Coefficients

5.4.1. Binomial Theorem. The following identities can be easily checked:

$$\begin{aligned}(x+y)^0 &= 1\\(x+y)^1 &= x+y\\(x+y)^2 &= x^2+2\,xy+y^2\\(x+y)^3 &= x^3+3\,x^2y+3\,xy^2+y^3\end{aligned}$$

They can be generalized by the following formula, called the *Binomial Theorem*:

$$\begin{aligned}(x+y)^n &= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\\&= \binom{n}{0}x^n+\binom{n}{1}x^{n-1}y+\binom{n}{2}x^{n-2}y^2+\cdots\\&\qquad\qquad+\binom{n}{n-1}xy^{n-1}+\binom{n}{n}y^n\,.\end{aligned}$$

We can find this formula by writing

$$(x+y)^n=(x+y)\times(x+y)\times\overset{(n\text{ factors})}{\cdots}\times(x+y)\,,$$

expanding, and grouping terms of the form x^ay^b. Since there are n factors of the form $(x+y)$, we have $a+b=n$, hence the terms must be of the form $x^{n-k}y^k$. The coefficient of $x^{n-k}y^k$ will be equal to the number of ways in which we can select the y from any k of the factors (and the x from the remaining $n-k$ factors), which is $C(n,k)=\binom{n}{k}$. The expression $\binom{n}{k}$ is often called *binomial coefficient*.

Exercise: Prove

$$\sum_{k=0}^{n}\binom{n}{k}=2^n\qquad\text{and}\qquad\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0\,.$$

Hint: Apply the binomial theorem to $(1+1)^2$ and $(1-1)^2$.

5.4.2. Properties of Binomial Coefficients. The binomial coefficients have the following properties:

1. $\binom{n}{k}=\binom{n}{n-k}$

2. $\displaystyle\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$

The first property follows easily from $\displaystyle\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

The second property can be proved by choosing a distinguished element a in a set A of $n+1$ elements. The set A has $\binom{n+1}{k+1}$ subsets of size $k+1$. Those subsets can be partitioned into two classes: that of the subsets containing a, and that of the subsets not containing a. The number of subsets containing a equals the number of subsets of $A - \{a\}$ of size k, i.e., $\binom{n}{k}$. The number of subsets not containing a is the number of subsets of $A - \{a\}$ of size $k+1$, i.e., $\binom{n}{k+1}$. Using the sum principle we find that in fact $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$.

5.4.3. Pascal's Triangle. The properties shown in the previous section allow us to compute binomial coefficients in a simple way. Look at the following triangular arrangement of binomial coefficients:

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}$$

We notice that each binomial coefficient on this arrangement must be the sum of the two closest binomial coefficients on the line above it. This together with $\binom{n}{0} = \binom{n}{n} = 1$, allows us to compute very quickly the values of the binomial coefficients on the arrangement:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

This arrangement of binomial coefficients is called *Pascal's Triangle*.[1]

[1] Although it was already known by the Chinese in the XIV century.

5.5. The Pigeonhole Principle

5.5.1. The Pigeonhole Principle. The *pigeonhole principle* is used for proving that a certain situation must actually occur. It says the following: *If n pigeonholes are occupied by m pigeons and $m > n$, then at least one pigeonhole is occupied by more than one pigeon.*[1]

Example: In any given set of 13 people at least two of them have their birthday during the same month.

Example: Let S be a set of eleven 2-digit numbers. Prove that S must have two elements whose digits have the same difference (for instance in $S = \{10, 14, 19, 22, 26, 28, 49, 53, 70, 90, 93\}$, the digits of the numbers 28 and 93 have the same difference: $8 - 2 = 6$, $9 - 3 = 6$.) *Answer*: The digits of a two-digit number can have 10 possible differences (from 0 to 9). So, in a list of 11 numbers there must be two with the same difference.

Example: Assume that we choose three different digits from 1 to 9 and write all permutations of those digits. Prove that among the 3-digit numbers written that way there are two whose difference is a multiple of 500. *Answer*: There are $9 \cdot 8 \cdot 7 = 504$ permutations of three digits. On the other hand if we divide the 504 numbers by 500 we can get only 500 possible remainders, so at least two numbers give the same remainder, and their difference must be a multiple of 500.

Exercise: Prove that if we select $n + 1$ numbers from the set $S = \{1, 2, 3, \ldots, 2n\}$, among the numbers selected there are two such that one is a multiple of the other one.

[1] The *Pigeonhole Principle* (*Schubfachprinzip*) was first used by Dirichlet in Number Theory. The term *pigeonhole* actually refers to one of those old-fashioned writing desks with thin vertical wooden partitions in which to file letters.