CHAPTER 7

Graph Theory

7.1. Graphs

7.1.1. Graphs. Consider the following examples:

1. A road map, consisting of a number of towns connected with roads.
2. The representation of a binary relation defined on a given set. The relation of a given element x to another element y is represented with an arrow connecting x to y.

The former is an example of (undirected) *graph*. The latter is an example of a *directed graph* or *digraph*.

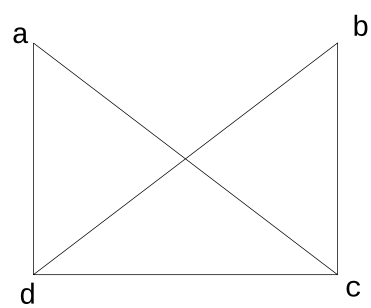

FIGURE 7.1. Undirected Graph.

In general a *graph* G consists of two things:

1. The *vertex set* V, whose elements are called *vertices*, *nodes* or *points*.
2. The *edge set* E or set of *edges* connecting pairs of vertices. If the edges are directed then they are also called *directed edges* or *arcs*. Each edge $e \in E$ is associated with a pair of vertices.

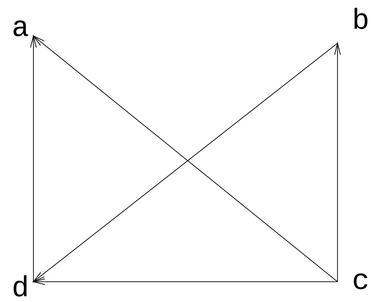

FIGURE 7.2. Directed Graph.

A graph is sometimes represented by the pair (V, E) (we assume V and E finite).

If the graph is undirected and there is a unique edge e connecting x and y we may write $e = \{x, y\}$, so E can be regarded as set of unordered pairs. In this context we may also write $e = (x, y)$, understanding that here (x, y) is not an ordered pair, but the name of an edge.

If the graph is directed and there is a unique edge e pointing from x to y, then we may write $e = (x, y)$, so E may be regarded as a set of ordered pairs. If $e = (x, y)$, the vertex x is called *origin*, *source* or *initial point* of the edge e, and y is called the *terminus*, *terminating vertex* or *terminal point*.

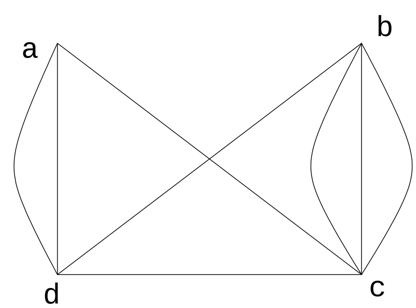

FIGURE 7.3. Graph with parallel edges.

Two vertices connected by an edge are called *adjacent*. They are also the *endpoints* of the edge, and the edge is said to be *incident* to each of its endpoints. If the graph is directed, an edge pointing from vertex x to vertex y is said to be *incident from* x and *incident to* y. An edge connecting a vertex to itself is called a *loop*. Two edges connecting the same pair of points (and pointing in the same direction if the graph is directed) are called *parallel* or *multiple*.

A graph with neither loops nor multiple edges is called a *simple graph*. If a graph has multiple edges but no loops then it is called a *multigraph*. If it has loops (and possible also multiple edges) then it is called a *pseudograph*.

The following table summarizes the graph terminology

TABLE 7.1.1. Graph Terminology

Type	**Edges**	**Multiple Edges Allowed?**	**Loops Allowed?**
Simple graph	indirected	no	no
Multigraph	indirected	yes	no
Pseudograph	indirected	yes	yes
Directed graph	directed	no	yes
Directed multigraph	directed	yes	yes

The *degree* of a vertex v, represented $\deg(v)$, is the number of edges that contain it (loops are counted twice). A vertex of degree zero (not connected to any other vertex) is called *isolated*. A vertex of degree 1 is called *pendant*.

The Handshaking Theorem. Let $G = (V, E)$ be an undirected graph with e edges. Then

$$2e = \sum_{v \in V} \deg(v) .$$

(This applies even if multiple edges and loops are present.)

In a graph with directed edges, the *in-degree* of a vertex v, denoted $\deg^-(v)$, is the number of edges with v as their terminal vertex. The *out-degree* of a vertex v, denoted $\deg^+(v)$, is the number of edges with v as their initial vertex. (Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.)

Number of vertices of odd degree. An undirected graph has an even number of vertices of odd degree. *Proof:* Let V_e and V_o respectively the set of vertices of even degree and the set of vertices of odd degree in an undirected graph $G = (V, E)$. Then

$$2e = \sum_{v \in V} \deg(v) = \sum_{v \in V_e} \deg(v) + \sum_{v \in V_o} \deg(v) .$$

Since $\deg(v)$ is even for $v \in V_e$, the first sum in the right hand side of the equality is even. The total sum must be $2e$, which is even, so the second sum must be even too. But its terms are all odd, so there must be an even number of them.

Sum of degrees in an directed graph. Let $G = (V, E)$ be a directed graph. Then

$$\sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} \deg^{+}(v) = |E| .$$

A *weighted graph* is a graph whose edges have been labeled with numbers. The *length* of a path in a weighted graph is the sum of the weights of the edges in the path.

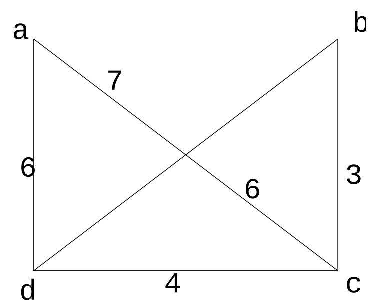

FIGURE 7.4. Weighted Graph.

7.1.2. Special Graphs. Here we examine a few special graphs.

The n-cube: A graph with with 2^n vertices labeled $0, 1, \ldots, 2^n - 1$ so that two of them are connected with an edge if their binary representation differs in exactly one bit.

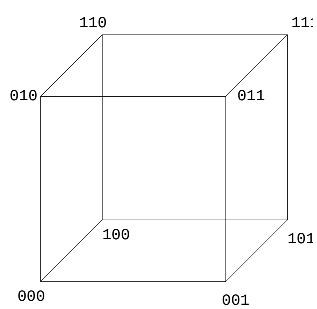

FIGURE 7.5. 3-cube.

Complete Graph: a simple undirected graph G such that every pair of distinct vertices in G are connected by an edge. The *complete graph* of n vertices is represented K_n (fig. 7.6). A *complete directed graph* is a simple directed graph $G = (V, E)$ such that every pair of distinct vertices in G are connected by exactly one edge—so, for each pair of distinct vertices, either (x, y) or (y, x) (but not both) is in E.

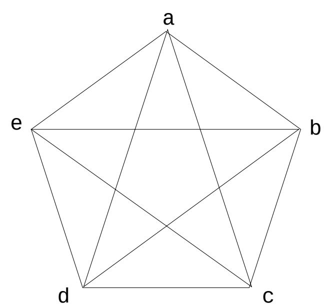

FIGURE 7.6. Complete graph K_5.

Bipartite Graph: a graph $G = (V, E)$ in which V can be partitioned into two subsets V_1 and V_2 so that each edge in G connects some vertex in V_1 to some vertex in V_2. A bipartite simple graph is called *complete* if each vertex in V_1 is connected to each vertex in V_2. If $|V_1| = m$ and $|V_2| = n$, the corresponding complete bipartite graph is represented $K_{m,n}$ (fig. 7.7).

A graph is bipartite iff its vertices can be colored with two colors so that every edge connects vertices of different color.

Question: Is the n-cube bipartite. Hint: color in red all vertices whose binary representation has an even number of 1's, color in blue the ones with an odd number of 1's.

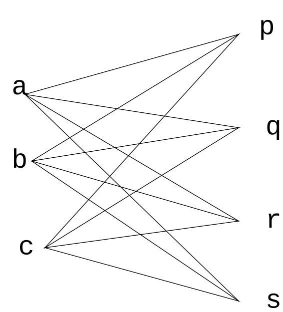

FIGURE 7.7. Complete bipartite graph $K_{3,4}$.

Regular Graph: a simple graph whose vertices have all the same degree. For instance, the n-cube is regular.

7.1.3. Subgraph. Given a graph $G = (V, E)$, a *subgraph* $G' = (V', E')$ of G is another graph such that $V' \subseteq V$ and $E' \subseteq E$. If $V' = V$ then G' is called a *spanning subgraph* of G.

Given a subset of vertices $U \subseteq V$, the subgraph of G *induced* by U, denoted $\langle U \rangle$, is the graph whose vertex set is U, and its edge set contains all edges from G connecting vertices in U.