

## CHAPTER 1

# Logic, Proofs

### 1.1. Propositions

A *proposition* is a declarative sentence that is either true or false (but not both). For instance, the following are propositions: “Paris is in France” (true), “London is in Denmark” (false), “ $2 < 4$ ” (true), “ $4 = 7$  (false)”. However the following are not propositions: “what is your name?” (this is a question), “do your homework” (this is a command), “this sentence is false” (neither true nor false), “ $x$  is an even number” (it depends on what  $x$  represents), “Socrates” (it is not even a sentence). The truth or falsehood of a proposition is called its *truth value*.

**1.1.1. Connectives, Truth Tables.** *Connectives* are used for making *compound propositions*. The main ones are the following ( $p$  and  $q$  represent given propositions):

Name	Represented	Meaning
Negation	$\neg p$	“not $p$ ”
Conjunction	$p \wedge q$	“ $p$ and $q$ ”
Disjunction	$p \vee q$	“ $p$ or $q$ (or both)”
Exclusive Or	$p \oplus q$	“either $p$ or $q$ , but not both”
Implication	$p \rightarrow q$	“if $p$ then $q$ ”
Biconditional	$p \leftrightarrow q$	“ $p$ if and only if $q$ ”

The truth value of a compound proposition depends only on the value of its components. Writing F for “false” and T for “true”, we can summarize the meaning of the connectives in the following way:

$p$	$q$	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	T	T	F	T	T
T	F	F	F	T	T	F	F
F	T	T	F	T	T	T	F
F	F	T	F	F	F	T	T

Note that  $\vee$  represents a *non-exclusive* or, i.e.,  $p \vee q$  is true when any of  $p$ ,  $q$  is true and also when both are true. On the other hand  $\oplus$  represents an *exclusive* or, i.e.,  $p \oplus q$  is true only when exactly one of  $p$  and  $q$  is true.

### 1.1.2. Tautology, Contradiction, Contingency.

1. A proposition is said to be a *tautology* if its truth value is T for any assignment of truth values to its components. *Example:* The proposition  $p \vee \neg p$  is a tautology.
2. A proposition is said to be a *contradiction* if its truth value is F for any assignment of truth values to its components. *Example:* The proposition  $p \wedge \neg p$  is a contradiction.
3. A proposition that is neither a tautology nor a contradiction is called a *contingency*.

$p$	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
T	F	T	F
F	T	T	F
F	T	T	F

$\uparrow$                        $\uparrow$   
 tautology              contradiction

**1.1.3. Conditional Propositions.** A proposition of the form “if  $p$  then  $q$ ” or “ $p$  implies  $q$ ”, represented “ $p \rightarrow q$ ” is called a *conditional proposition*. For instance: “if John is from Chicago then John is from Illinois”. The proposition  $p$  is called *hypothesis* or *antecedent*, and the proposition  $q$  is the *conclusion* or *consequent*.

Note that  $p \rightarrow q$  is true always except when  $p$  is true and  $q$  is false. So, the following sentences are true: “if  $2 < 4$  then Paris is in France” (true  $\rightarrow$  true), “if London is in Denmark then  $2 < 4$ ” (false  $\rightarrow$  true),

“if  $4 = 7$  then London is in Denmark” ( $\text{false} \rightarrow \text{false}$ ). However the following one is false: “if  $2 < 4$  then London is in Denmark” ( $\text{true} \rightarrow \text{false}$ ).

It might seem strange that “ $p \rightarrow q$ ” is considered true when  $p$  is false, regardless of the truth value of  $q$ . This will become clearer when we study *predicates* such as “if  $x$  is a multiple of 4 then  $x$  is a multiple of 2”. That implication is obviously true, although for the particular case  $x = 3$  it becomes “if 3 is a multiple of 4 then 3 is a multiple of 2”.

The proposition  $p \leftrightarrow q$ , read “ $p$  if and only if  $q$ ”, is called *biconditional*. It is true precisely when  $p$  and  $q$  have the same truth value, i.e., they are both true or both false.

**1.1.4. Logical Equivalence.** Note that the compound propositions  $p \rightarrow q$  and  $\neg p \vee q$  have the same truth values:

$p$	$q$	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

When two compound propositions have the same truth values no matter what truth value their constituent propositions have, they are called *logically equivalent*. For instance  $p \rightarrow q$  and  $\neg p \vee q$  are logically equivalent, and we write it:

$$p \rightarrow q \equiv \neg p \vee q$$

Note that that two propositions  $A$  and  $B$  are logically equivalent precisely when  $A \leftrightarrow B$  is a tautology.

*Example:* De Morgan’s Laws for Logic. The following propositions are logically equivalent:

$$\begin{aligned} \neg(p \vee q) &\equiv \neg p \wedge \neg q \\ \neg(p \wedge q) &\equiv \neg p \vee \neg q \end{aligned}$$

We can check it by examining their truth tables:

$p$	$q$	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F	T	F	F	T	F	F
T	F	F	T	T	F	F	F	T	T
F	T	T	F	T	F	F	F	T	T
F	F	T	T	F	T	T	F	T	T

*Example:* The following propositions are logically equivalent:

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

Again, this can be checked with the truth tables:

$p$	$q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

*Exercise:* Check the following logical equivalences:

$$\begin{aligned} \neg(p \rightarrow q) &\equiv p \wedge \neg q \\ p \rightarrow q &\equiv \neg q \rightarrow \neg p \\ \neg(p \leftrightarrow q) &\equiv p \oplus q \end{aligned}$$

**1.1.5. Converse, Contrapositive.** The *converse* of a conditional proposition  $p \rightarrow q$  is the proposition  $q \rightarrow p$ . As we have seen, the biconditional proposition is equivalent to the conjunction of a conditional proposition and its converse.

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

So, for instance, saying that “John is married if and only if he has a spouse” is the same as saying “if John is married then he has a spouse” and “if he has a spouse then he is married”.

Note that the converse is not equivalent to the given conditional proposition, for instance “if John is from Chicago then John is from Illinois” is true, but the converse “if John is from Illinois then John is from Chicago” may be false.

The *contrapositive* of a conditional proposition  $p \rightarrow q$  is the proposition  $\neg q \rightarrow \neg p$ . They are logically equivalent. For instance the contrapositive of “if John is from Chicago then John is from Illinois” is “if John is not from Illinois then John is not from Chicago”.