MATH 214-2 - Fall 2001 - Final Exam (solutions)

SOLUTIONS

1. (Numerical Integration) Find the Trapezoidal (T_4), Midpoint (M_4) and Simpson's (S_4) approximations with 4 subintervals to the following integral:

$$\int_{-2}^{2} x^2\, dx\,.$$

Solution:

Note that in all cases $\Delta x = \dfrac{2-(-2)}{4} = 1$.

1. Trapezoidal approximation:

$$T_4 = \frac{\Delta x}{2} \cdot \{y_0^2 + 2y_1^2 + 2y_2^2 + 2y_3^2 + y_4^2\}$$
$$= \frac{1}{2} \cdot \{(-2)^2 + 2\cdot(-1)^2 + 2\cdot 0^2 + 2\cdot 1^2 + 2^2\} = \boxed{6}$$

2. Midpoint approximation:

$$M_4 = \Delta x \cdot \{y_{1/2}^2 + y_{3/2}^2 + y_{5/2}^2 + y_{7/2}^2\}$$
$$= 1 \cdot \{(-1.5)^2 + (-0.5)^2 + 0.5^2 + 1.5^2\} = \boxed{5}$$

3. Simpson's approximation:

$$S_4 = \frac{\Delta x}{3} \cdot \{y_0^2 + 4y_1^2 + 2y_2^2 + 4y_3^2 + y_4^2\}$$
$$= \frac{1}{3} \cdot \{(-2)^2 + 4\cdot(-1)^2 + 2\cdot 0^2 + 4\cdot 1^2 + 2^2\} = \boxed{\frac{16}{3}}$$

2. (Volumes of solids) Find the volume of the solid obtained by rotating around the y-axis the area under $y = \sin x$ from $x = 0$ to $x = \pi$.

Solution:

We use the method of cylindrical shells:

$$V = \int_0^\pi 2\pi x y \, dx = 2\pi \int_0^\pi x \sin x \, dx \, .$$

The integral can be evaluated by parts, using $u = x$, $v = -\cos x$:

$$\int x \sin x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + C \, ,$$

Hence:

$$V = 2\pi \Big[-x \cos x + \sin x \Big]_0^\pi = 2\pi(-\pi \cos \pi) = \boxed{2\pi^2}$$

3. (Surface Areas) Find the area of the surface obtained by revolving the curve $y = \frac{2}{3}x^{3/2}$, $3 \leq x \leq 8$, around the x-axis—just setup the integral, do not try to evaluate it.

Solution:

$$A = \int_{x=3}^{x=8} 2\pi y \, ds = \int_3^8 2\pi y \sqrt{1 + (y')^2} \, dx = \boxed{\frac{4\pi}{3} \int_3^8 x^{3/2} \sqrt{1 + x} \, dx}$$

4. (Separable Differential Equations) Solve the following initial value problem:

$$\begin{cases} \dfrac{dx}{dt} = x\,(1-x) \\ x(0) = \frac{1}{2} \end{cases}$$

Solution:

Separating variables we get:

$$\frac{dx}{x\,(1-x)} = dt\,.$$

The left hand side can be integrated in the following way:

$$\int \frac{dx}{x\,(1-x)} = \int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx = \ln x - \ln(1-x) + C'\,,$$

hence:

$$\ln x - \ln(1-x) = t + C\,.$$

According to the initial condition we have:

$$\ln(1/2) - \ln(1/2) = C \quad \Rightarrow \quad C = 0\,.$$

So the solution is

$$\ln x - \ln(1-x) = t \quad \Rightarrow \quad \frac{x}{1-x} = e^t\,.$$

Solving for x we get:

$$\boxed{x(t) = \frac{e^t}{1+e^t}}$$

5. (Logarithmic Differentiation) Use *logarithmic differentiation* to find the derivative of

$$y = \frac{\sqrt{1+x^2}\,\sqrt[4]{1+x^4}}{\sqrt[3]{1+x^3}\,\sqrt[5]{1+x^5}}$$

Solution:

First we take logarithms and simplify:

$$\ln y = \frac{1}{2}\ln\left(1+x^2\right) - \frac{1}{3}\ln\left(1+x^3\right) + \frac{1}{4}\ln\left(1+x^4\right) - \frac{1}{5}\ln\left(1+x^5\right).$$

Next we differentiate:

$$\frac{y'}{y} = \frac{x}{1+x^2} - \frac{x^2}{1+x^3} + \frac{x^3}{1+x^4} - \frac{x^4}{1+x^5}.$$

Hence:

$$\boxed{y' = \frac{\sqrt{1+x^2}\,\sqrt[4]{1+x^4}}{\sqrt[3]{1+x^3}\,\sqrt[5]{1+x^5}}\left(\frac{x}{1+x^2} - \frac{x^2}{1+x^3} + \frac{x^3}{1+x^4} - \frac{x^4}{1+x^5}\right)}$$

6. (L'Hôpital's Rule) Find the following limits:

1. $\lim\limits_{x\to\infty} \dfrac{x^2-1}{4x^2-x}$
2. $\lim\limits_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{\ln(1+x)}\right)$
3. $\lim\limits_{x\to 1} x^{1/(1-x)}$

Solution:

1. $\lim\limits_{x\to\infty} \dfrac{x^2-1}{4x^2-x} = \lim\limits_{x\to\infty} \dfrac{2x}{8x-1} = \lim\limits_{x\to\infty} \dfrac{2}{8} = \boxed{\dfrac{1}{4}}$
2. $\lim\limits_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{\ln(1+x)}\right) = \lim\limits_{x\to 0} \dfrac{\ln(1+x)-x}{x\ln(1+x)} = \lim\limits_{x\to 0} \dfrac{\frac{1}{(1+x)}-1}{\ln(1+x)+\frac{x}{1+x}}$

   $= \lim\limits_{x\to 0} \dfrac{-x}{(1+x)\ln(1+x)+x} = \lim\limits_{x\to 0} \dfrac{-1}{\ln(1+x)+1+1} = \boxed{-\dfrac{1}{2}}$
3. If $L = \lim\limits_{x\to 1} x^{1/(1-x)}$, then

   $\ln(L) = \lim\limits_{x\to 1} \frac{1}{1-x}\ln x = \lim\limits_{x\to 1} \dfrac{\ln x}{1-x} = \lim\limits_{x\to 1} \dfrac{1/x}{-1} = -1\,,$

   hence

   $$\boxed{L = e^{-1}}$$

7. (Integration by Parts) Find the following integral using *integration by parts*:

$$\int \ln(1+x^2)\,dx =$$

Solution:

We make $u = \ln(1+x^2)$, $dv = dx$, so $du = \dfrac{2x\,dx}{1+x^2}$, $v = x$:

$$\begin{aligned}
\int \underbrace{\ln(1+x^2)}_{u}\ \underbrace{dx}_{dv} &= \int u\,dv = uv - \int v\,du \\
&= x\ln(1+x^2) - 2\int \frac{x^2}{1+x^2}\,dx \\
&= x\ln(1+x^2) - 2\int \left(1 - \frac{1}{1+x^2}\right)\,dx \\
&= \boxed{x\ln(1+x^2) - 2x + 2\tan^{-1}x + C}
\end{aligned}$$

8. (Partial Fractions) Find the following integral by decomposing the integrand into *partial fractions*:

$$\int \frac{x^2}{x^4-1}\,dx =$$

Solution:

First we factor the denominator:

$$x^4 - 1 = (x^2+1)(x^2-1) = (x^2+1)(x+1)(x-1)\,.$$

Next we decompose the integrand into partial fractions:

$$\frac{x^2}{x^4-1} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1}$$

$$\begin{aligned}
x^2 &= (Ax+B)(x+1)(x-1) + C(x^2+1)(x-1) + D(x^2+1)(x+1)\\
x = 1 \quad &\Rightarrow \quad 1 = 4D \quad \Rightarrow \quad D = \frac{1}{4}\\
x = -1 \quad &\Rightarrow \quad 1 = -4C \quad \Rightarrow \quad C = -\frac{1}{4}\\
x = 0 \quad &\Rightarrow \quad 0 = -B - C + D = -B + \frac{1}{4} + \frac{1}{4} \quad \Rightarrow \quad B = \frac{1}{2}\\
x = 2 \quad &\Rightarrow \quad 4 = (2A+B)\cdot 3 + 5C + 15D \quad \Rightarrow \quad 4 = \left(6A + \frac{3}{2}\right) + \frac{5}{2}\\
&\Rightarrow \quad A = 0
\end{aligned}$$

So:

$$\frac{x^2}{x^4-1} = \frac{1/2}{x^2+1} - \frac{1/4}{x+1} + \frac{1/4}{x-1}$$

Hence:

$$\begin{aligned}
\int \frac{x^2}{x^4-1}\,dx &= \frac{1}{2}\int \frac{1}{x^2+1}\,dx - \frac{1}{4}\int \frac{1}{x+1}\,dx + \frac{1}{4}\int \frac{1}{x-1}\,dx\\
&= \boxed{\frac{1}{2}\tan^{-1} x - \frac{1}{4}\ln|x+1| + \frac{1}{4}\ln|x-1| + C}
\end{aligned}$$

9. (Integrals Containing Quadratic Polynomials) Find the following integral:

$$\int \frac{1}{(x^2+2x+2)^2}\,dx =$$

Solution:

$$\begin{aligned}
\int \frac{1}{(x^2+2x+2)^2}\,dx &= \int \frac{1}{((x+1)^2+1)^2}\,dx \\
&= \int \frac{1}{(u^2+1)^2}\,du && (u = x+1) \\
&= \int \frac{1}{(\tan^2 t+1)^2}\,\sec^2 t\,dt && (u = \tan t) \\
&= \int \frac{1}{\sec^2 t}\,dt = \int \cos^2 t\,dt \\
&= \frac{1}{2}\cos t\sin t + \frac{1}{2}\int 1\,dt + C && \text{(reduction formula)} \\
&= \frac{1}{2}\cos t\sin t + \frac{1}{2}t + C \\
&= \frac{1}{2}\frac{\tan t}{1+\tan^2 t} + \frac{1}{2}t + C \\
&= \frac{1}{2}\frac{u}{1+u^2} + \frac{1}{2}\tan^{-1} u + C \\
&= \boxed{\frac{1}{2}\frac{x+1}{x^2+2x+2} + \frac{1}{2}\tan^{-1}(x+1) + C}
\end{aligned}$$

10. (Taylor Series and Polynomials) Find the sixth degree Taylor polynomial of $f(x) = \sin^2 x$ at 0.

Solution:

We have:

$$
\begin{aligned}
f^{(0)}(x) &= \sin^2 x & f^{(0)}(0) &= 0 \\
f^{(1)}(x) &= 2\sin x \cos x & f^{(1)}(0) &= 0 \\
f^{(2)}(x) &= 2\cos^2 x - 2\sin^2 x & f^{(2)}(0) &= 2 \\
f^{(3)}(x) &= -8\sin x \cos x & f^{(3)}(0) &= 0 \\
f^{(4)}(x) &= -8\cos^2 x + 8\sin^2 x & f^{(4)}(0) &= -8 \\
f^{(5)}(x) &= 32\sin x \cos x & f^{(5)}(0) &= 0 \\
f^{(6)}(x) &= 32\cos^2 x - 32\sin^2 x & f^{(6)}(0) &= 32
\end{aligned}
$$

Hence:

$$P_6(x) = \sum_{n=0}^{6} \frac{f^{(n)}}{n!} x^n = \frac{2}{2!}x^2 + \frac{-8}{4!}x^4 + \frac{32}{6!}x^6 = \boxed{x^2 - \frac{x^4}{3} + \frac{2x^6}{45}}$$

Alternatively we may use the half-angle trigonometric identity and the Maclaurin series for $\cos x$:

$$
\begin{aligned}
\sin^2 x &= \frac{1}{2}(1 - \cos 2x) = \frac{1}{2}\left\{1 - \left(1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} - \frac{64x^6}{6!} + \dots\right)\right\} \\
&= x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \dots
\end{aligned}
$$

and then truncate at the sixth degree term.

A third method would be to expand the square of the Maclaurin series for $\sin x$:

$$\sin^2 x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)^2$$

and also truncate at the sixth degree term.

Table of Integrals

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C \ (n \neq -1) \qquad \int \frac{du}{u} = \ln|u| + C$$

$$\int e^u\,du = e^u + C \qquad \int \cos u\,du = \sin u + C$$

$$\int \sin u\,du = -\cos u + C \qquad \int \sec^2 u\,du = \tan u + C$$

$$\int \csc^2 u\,du = -\cot u + C \qquad \int \sec u \tan u\,du = \sec u + C$$

$$\int \csc u \cot u\,du = -\csc u + C \qquad \int \sec u\,du = \ln|\sec u + \tan u| + C$$

$$\int \csc u\,du = \ln|\csc u - \cot u| + C \qquad \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$$

$$\int \frac{du}{1+u^2} = \tan^{-1} u + C \qquad \int \frac{du}{u\sqrt{u^2-1}}\,du = \sec^{-1}|u| + C$$

Integrals Involving Inverse Hyperbolic Functions

$$\int \frac{du}{\sqrt{u^2+1}} = \sinh^{-1} u + C \qquad \int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + C$$

$$\int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1}|u| + C \qquad \int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1}|u| + C$$

Reduction Formulas

$$\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1} u\,\cos u + \frac{n-1}{n}\int \sin^{n-2} u\,du$$

$$\int \cos^n u\,du = \frac{1}{n}\cos^{n-1} u\,\sin u + \frac{n-1}{n}\int \cos^{n-2} u\,du$$

$$\int \tan^n u\,du = \frac{\tan^{n-1} u}{n-1} - \int \tan^{n-2} u\,du\,.$$

$$\int \sec^n u\,du = \frac{\sec^{n-2} u \tan u}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} u\,du\,.$$