1.10. Improper Integrals

1.10.1. Improper Integrals. Up to now we have studied integrals of the form

$$\int_a^b f(x)\,dx\,,$$

where f is a *continuous* function defined on the *closed* and *bounded* interval $[a,b]$. *Improper integrals* are integrals in which one or both of these conditions are not met, i.e.,

(1) The interval of integration is not bounded:

$$[a,+\infty)\,,\qquad (-\infty,a]\,,\qquad (-\infty,+\infty)\,,$$

e.g.:

$$\int_1^\infty \frac{1}{x^2}\,dx\,.$$

(2) The integrand has an infinite discontinuity at some point c in $[a,b]$:

$$\lim_{x\to c} f(x) = \pm\infty\,.$$

e.g.:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx\,.$$

1.10.2. Infinite Limits of Integration. Improper Integrals of Type 1. In case one of the limits of integration is infinite, we define:

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx\,,$$

or

$$\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty}\int_t^a f(x)\,dx\,.$$

If both limits of integration are infinite, then we choose any c and define:

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx\,.$$

If the limits defining the integral exist the integral is called *convergent*, otherwise it is called *divergent*.

Remark: Sometimes we write $[F(x)]_a^\infty$ as an abbreviation for

$$[F(x)]_a^\infty = \lim_{t\to\infty}\,[F(x)]_a^t\,.$$

Analogously:

$$[F(x)]_{-\infty}^{a} = \lim_{t\to-\infty} [F(x)]_{t}^{a} \ ,$$

and

$$[F(x)]_{-\infty}^{\infty} = [F(x)]_{-\infty}^{c} + [F(x)]_{c}^{\infty} = \lim_{t\to-\infty} [F(x)]_{t}^{c} + \lim_{t\to\infty} [F(x)]_{c}^{t} \ .$$

Example:

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty} \left(-\frac{1}{t}+1\right) = 1\,,$$

or in simplified notation:

$$\int_1^\infty \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^\infty = \lim_{t\to\infty} \left(-\frac{1}{t}+1\right) = 1\,.$$

Example: For what values of p is the following integral convergent?:

$$\int_1^\infty \frac{1}{x^p}\,dx\,.$$

Answer: If $p = 1$ then we have

$$\int_1^t \frac{1}{x}\,dx = [\ln x]_1^t = \ln t\,,$$

so

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx = \lim_{t\to\infty} \ln t = \infty\,,$$

and the integral is divergent. Now suppose $p \neq 1$:

$$\int_1^t \frac{1}{x^p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \frac{1}{1-p}\left\{\frac{1}{t^{p-1}}-1\right\}$$

If $p > 1$ then $p - 1 > 0$ and

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \frac{1}{1-p}\left\{\frac{1}{t^{p-1}}-1\right\} = 0\,,$$

hence the integral is convergent. On the other hand if $p < 1$ then $p - 1 < 0$, $1 - p > 0$ and

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \frac{1}{1-p}\left\{t^{1-p}-1\right\} = \infty\,,$$

hence the integral is divergent. So:

$$\int_1^\infty \frac{1}{x^p}\,dx \ \text{ is convergent if } p > 1 \text{ and divergent if } p \leq 1.$$

1.10.3. Infinite Integrands. Improper Integrals of Type 2. Assume f is defined in $[a, b)$ but

$$\lim_{x \to b^-} f(x) = \pm\infty \,.$$

Then we define

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx \,.$$

Analogously, if f is defined in $(a, b]$ but

$$\lim_{x \to a^+} f(x) = \pm\infty \,.$$

Then we define

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx \,.$$

Finally, if $f(x)$ has an infinite discontinuity at c inside $[a, b]$, then the definition is

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \,.$$

If the limits defining the integral exist the integral is called *convergent*, otherwise it is called *divergent*.

Remark: If the interval of integration is $[a, b)$ sometimes we write $[F(x)]_a^b$ as an abbreviation for $\lim_{t \to b^-} [F(x)]_a^t$—and analogously for intervals of the form $(a, b]$.

Example:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t \to 0^-} \int_t^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t \to 0^-} \left[2\sqrt{x}\right]_t^1 = \lim_{t \to 0^-} \left(2 - 2\sqrt{t}\right) = 2 \,,$$

or in simplified notation:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_0^1 = \lim_{t \to 0^-} \left(2 - 2\sqrt{t}\right) = 2 \,.$$

Example: Evaluate $\displaystyle\int_0^1 \ln x\,dx$.

Answer: The function $\ln x$ has a vertical asymptote at $x = 0$ because $\lim\limits_{x\to 0^+} \ln x = -\infty$. Hence:

$$\begin{aligned}
\int_0^1 \ln x\, dx &= \lim_{t\to 0^+} \int_t^1 \ln x\, dx \\
&= \lim_{t\to 0^+} [x \ln x - x]_t^1 \\
&= \lim_{t\to 0^+} \{(1 \ln 1 - 1) - (t \ln t - t)\} \\
&= \lim_{t\to 0^+} \{t - 1 - t \ln t\} \qquad\qquad (\lim_{t\to 0^+} t \ln t = 0) \\
&= \boxed{-1}\,.
\end{aligned}$$

1.10.4. Comparison Test for Improper Integrals. Suppose f and g are continuous functions such that $f(x) \geq g(x) \geq 0$ for $x \geq 0$.

(1) If $\int_a^\infty f(x)\, dx$ if convergent then $\int_a^\infty g(x)\, dx$ is convergent.
(2) If $\int_a^\infty g(x)\, dx$ if divergent then $\int_a^\infty f(x)\, dx$ is divergent.

A similar statement holds for type 2 integrals.

Example: Prove that $\displaystyle\int_0^\infty e^{-x^2}\, dx$ is convergent.

Answer: We have:

$$\int_0^\infty e^{-x^2}\, dx = \int_0^1 e^{-x^2}\, dx + \int_1^\infty e^{-x^2}\, dx\,.$$

The first integral on the right hand side is an ordinary definite integral so we only need to show that the second integral is convergent. In fact, for $x \geq 1$ we have $x^2 \geq x$, so $e^{-x^2} \leq e^{-x}$. On the other hand:

$$\int_1^t e^{-x}\, dx = \left[-e^{-x}\right]_1^t = -e^{-t} + e^{-1}\,,$$

hence

$$\int_1^\infty e^{-x}\, dx = \lim_{t\to\infty} \left(-e^{-t} + e^{-1}\right) = e^{-1}\,,$$

so $\int_1^\infty e^{-x}\, dx$ is convergent. Hence, by the comparison theorem $\int_1^\infty e^{-x^2}\, dx$ is convergent, QED.