PUTNAM TRAINING, 2008
COMPLEX NUMBERS

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Remark. This is a list of exercises on Complex Numbers —Miguel A. Lerma

Exercises

1. Let m and n two integers such that each can be expressed as the sum of two perfect squares. Prove that mn has this property as well. For instance $17 = 4^2 + 1^2$, $13 = 2^2 + 3^2$, and $17 \cdot 13 = 221 = 14^2 + 5^2$.

2. Prove that $\displaystyle\sum_{k=0}^{n} \sin k = \frac{\sin \frac{n}{2} \sin \frac{n+1}{2}}{\sin \frac{1}{2}}$.

3. Show that if z is a complex number such that $z + 1/z = 2\cos a$, then for any integer n, $z^n + 1/z^n = 2\cos na$.

4. Factor $p(z) = z^5 + z + 1$.

5. Find a close-form expression for $\displaystyle\prod_{k=1}^{n-1} \sin \frac{k\pi}{n}$.

6. Consider a regular n-gon which is inscribed in a circle with radius 1. What is the product of the lengths of all $n(n-1)/2$ diagonals of the polygon (this includes the sides of the n-gon).

7. (Putnam 1991, B2) Suppose f and g are non-constant, differentiable, real-valued functions on $\mathbb{R}$. Furthermore, suppose that for each pair of real numbers x and y

$$f(x+y) = f(x)\,f(y) - g(x)\,g(y)$$
$$g(x+y) = f(x)\,g(y) + g(x)\,f(y)$$

If $f'(0) = 0$ prove that $f(x)^2 + g(x)^2 = 1$ for all x.

8. Given a circle of n lights, exactly one of which is initially on, it is permitted to change the state of a bulb provided that one also changes the state of every dth bulb after it (where d is a divisor of n strictly less than n), provided that all n/d bulbs were originally in the same state as one another. For what values of n is it possible to turn all the bulbs on by making a sequence of moves of this kind?

9. Suppose that a, b, u, v are real numbers for which $av - bu = 1$. Prove that $a^2 + b^2 + u^2 + v^2 + au + bv \geq \sqrt{3}$.

10. Let P_n be a regular polygon inscribed in a unit circle. Denote by S_n the set of all chords whose endpoints are vertices of P_n, and let A_n be the average length of all chords in S_n. Write a closed form expression for A_n in terms of n, and compute $\lim_{n\to\infty} A_n$.

Hints

1. If $m = a^2 + b^2$ and $n = c^2 + d^2$, then consider the product $z = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$.

2. The left hand side of the equality is the imaginary part of $\sum_{k=0}^{n} e^{ik}$.

3. What are the possible values of z?

4. If $\omega = e^{2\pi i/3}$ then ω and ω^2 are two roots of $p(z)$.

5. Write $\sin t = (e^{ti} - e^{-ti})/2i$.

6. Assume the vertices of the n-gon placed on the complex plane at the nth roots of unity.

7. Look at the function $h(x) = f(x) + ig(x)$.

8. Assume the lights placed on the complex plane at the nth roots of unity $1, \zeta, \zeta^2, \dots, \zeta^{n-1}$, where $\zeta = e^{2\pi i/n}$.

9. Let $z_1 = a - bi$, $z_2 = u + vi$. We have $|z_1|^2 = a^2 + b^2$, $|z_2| = u^2 + v^2$, $\Re(z_1 z_2) = au + bv$, $\Im(z_1 z_2) = 1$, and must prove $|z_1|^2 + |z_2|^2 + \Re(z_1 z_2) \geq \sqrt{3}$.

10. The length of a chord can be expressed as $2 \sin \frac{\pi k}{n} = 2\Im\left\{e^{\frac{\pi k}{n} i}\right\}$, where k is an integer, $0 \leq k \leq n - 1$, and $\Im\{z\}$ = imaginary part of the complex number z.

Solutions

1. If $m = a^2 + b^2$ and $n = c^2 + d^2$, then consider the product $z = (a+bi)(c+di) = (ac-bd)+(ad+bc)i$. We have

$$|z|^2 = |a+bi|^2|c+di|^2 = (a^2+b^2)(c^2+d^2) = mn\,,$$

and

$$|z|^2 = (ac-bd)^2 + (ad+bc)^2\,,$$

so mn is also in fact a sum of two perfect squares.

2. The left hand side of the equality is the imaginary part of

$$\sum_{k=0}^{n} e^{ik} = \frac{e^{i(n+1)}-1}{e^i-1} = \frac{e^{i(n+1/2)}-e^{-i/2}}{e^{i/2}-e^{-i/2}} = \frac{\cos\left(n+\frac{1}{2}\right)-\cos\frac{1}{2}+i\{\sin\left(n+\frac{1}{2}\right)+\sin\frac{1}{2}\}}{2i\sin\frac{1}{2}}\,.$$

The imaginary part of that expression is

$$\frac{\cos\frac{1}{2}-\cos\left(n+\frac{1}{2}\right)}{2\sin\frac{1}{2}} = \frac{\sin\frac{n}{2}\sin\frac{n+1}{2}}{\sin\frac{1}{2}}$$

3. We have that $z = e^{\pm ia}$, so $z + 1/z = e^{ia} + e^{-ia} = 2\cos a$, hence:

$$z^n + 1/z^n = e^{ina} + e^{-ina} = 2\cos na\,.$$

4. Factoring a polynomial is easier to accomplish if we can find its roots. In this case we will look for roots that are roots of unity $e^{2k\pi i/n}$:

$$p(e^{2k\pi i/n}) = e^{10k\pi i/n} + e^{2k\pi i/n} + 1\,.$$

The three terms of that expression are complex numbers placed on the unit circle at the vertices of an equilateral triangle for $n = 3$ and $k = 1, 2$, so if $\omega = e^{2k\pi i/3}$, then ω and ω^2 are roots of $p(z)$, hence $p(z)$ is divisible by $(z-\omega)(z-\omega^2) = z^2+z+1$. By long division we find that the other factor is $z^3 - z^2 + 1$, hence:

$$p(z) = (z^2+z+1)(z^3-z^2+1)\,.$$

5. Write $\sin t = (e^{ti} - e^{-ti})/2i$ and consider the polynomial

$$p(x) = \prod_{k=1}^{n-1}(x - e^{2\pi ik/n})\,.$$

We have:

$$P = \prod_{k=1}^{n-1}\sin\frac{k\pi}{n} = \prod_{k=1}^{n-1}\frac{e^{\pi ik/n}-e^{-\pi ik/n}}{2i} = \frac{e^{-\pi i(n-1)/2}}{(2i)^{n-1}}\prod_{k=1}^{n-1}(e^{2\pi ik/n}-1) = \frac{p(1)}{2^{n-1}}\,.$$

On the other hand the roots of $p(x)$ are all nth roots of 1 except 1, so $(x-1)p(x) = x^n - 1$, and

$$p(x) = \frac{x^n-1}{x-1} = 1 + x + x^2 + \cdots + x^{n-1}\,.$$

Consequently $p(1) = n$, and $P = \dfrac{n}{2^{n-1}}$.

6. Assume the vertices of the n-gon placed on the complex plane at the nth roots of unity $1, \zeta, \zeta^2, \dots, \zeta^{n-1}$, where $\zeta = e^{2\pi i/n}$. Then the length of the diagonal connecting vertices j and k is $|\zeta^i - \zeta^k|$, and the desired product can be written

$$P = \prod_{0 \le j < k < n} |\zeta^j - \zeta^k| \,.$$

By symmetry we obtain the same product if we replace the condition $j < k$ with $k < j$, and multiplying both expressions together we get:

$$P^2 = \prod_{\substack{0 \le j,k < n \\ j \ne k}} |\zeta^j - \zeta^k| = |\zeta^j| \prod_{\substack{0 \le j,k < n \\ j \ne k}} |1 - \zeta^{k-j}| \,.$$

Note that $|\zeta^j| = 1$, and for each k, $r = k - j$ takes all non-zero values from $k - n + 1$ to k. Since $\zeta^r = \zeta^{r+n}$ we may assume that r ranges from 1 to $n-1$, so we can rewrite the product like this:

$$P^2 = \left(\prod_{r=1}^{n-1} |1 - \zeta^r| \right)^n \,.$$

Next consider the polynomial

$$p(x) = \prod_{r=1}^{n-1} (x - \zeta^r) \,.$$

Its roots are the same roots of $x^n - 1$ except 1, hence $x^n - 1 = (x-1)p(x)$ and

$$p(x) = \frac{x^n - 1}{x - 1} = 1 + x + x^2 + \cdots + x^{n-1} \,,$$

hence

$$\prod_{r=1}^{n-1} (1 - \zeta^r) = p(1) = n \,.$$

consequently $P^2 = n^n$, and $P = n^{n/2}$.

7. Define $h(x) = f(x) + ig(x)$. Then h is differentiable and $h'(0) = bi$ for some $b \in \mathbb{R}$. The given equations can be reinterpret as $h(x+y) = h(x)h(y)$. Differentiating respect to y and substituting $y = 0$ we get $h'(x) = h(x)h'(0) = bi \cdot h(x)$, so $h(x) = Ce^{bix}$ for some $C \in \mathbb{C}$. From $h(0+0) = h(0)h(0)$ we get $C = C^2$. If $C = 0$ then $h = 0$ and f and g would be constant, contradicting the hypothesis. Thus $C = 1$. Finally, for any $x \in \mathbb{R}$,

$$f(x)^2 + g(x)^2 = |h(x)|^2 = |e^{bix}|^2 = 1 \,.$$

8. Assume the lights placed on the complex plane at the nth roots of unity $1, \zeta, \zeta^2, \dots, \zeta^{n-1}$, where $\zeta = e^{2\pi i/n}$. Without loss of generality we may assume that the light at 1 is initially on. Now, if $d < n$ is a divisor of n and the lights $\zeta^a, \zeta^{a+d}, \zeta^{a+2d}, \dots, \zeta^{a+(\frac{n}{n}-1)d}$

have the same state, then we can change the state of this n/d lights. The sum of these is

$$\zeta^a + \zeta^{a+d} + \zeta^{a+2d} + \cdots + \zeta^{a+(\frac{n}{n}-1)d} = \zeta^a \left(\frac{1-\zeta^n}{1-\zeta^d} \right) = \zeta^a \left(\frac{1-1}{1-\zeta^d} \right) = 0\,.$$

So if we add up all the roots that are "on", the sum will never change. The original sum was 1, and the goal is to get all the lights turned on. That sum will be

$$1 + \zeta + \zeta^2 + \cdots + \zeta^{n-1} = \frac{1-\zeta^n}{1-\zeta} = 0 \neq 1\,.$$

Hence we can never turn on all the lights.

9. Let $z_1 = a - bi$, $z_2 = u + vi$. Then $|z_1|^2 = a^2 + b^2$, $|z_2| = u^2 + v^2$, $\Re(z_1 z_2) = au + bv$, $\Im(z_1 z_2) = 1$. On the other hand:

$$|z_1 z_2|^2 = \Re(z_1 z_2)^2 + \Im(z_1 z_2)^2 = \Re(z_1 z_2)^2 + 1\,.$$

Now for any real t,

$$(t\sqrt{3} + 1)^2 \geq 0 \implies 3t^2 + 1 \geq -2t\sqrt{3} \implies 4t^2 + 4 \geq (\sqrt{3} - t)^2\,.$$

Hence

$$(|z_1|^2 + |z_2|^2)^2 \geq 4|z_1 z_2|^2 = 4\big(\Re(z_1 z_2)^2 + 1\big) \geq \big(\sqrt{3} - \Re(z_1 z_2)\big)^2\,.$$

So, $|z_1|^2 + |z_2|^2 \geq \sqrt{3} - \Re(z_1 z_2)$. Or $|z_1|^2 + |z_2|^2 + \Re(z_1 z_2) \geq \sqrt{3}$, as required.

10. Name the vertices consecutively $V_0, \ldots, V_{n-1}$. The length of the chord joining vertices V_0 and V_k is

$$2 \sin \frac{\pi k}{n} = 2\Im \left\{ e^{\frac{\pi k}{n} i} \right\}\,,$$

where $\Im\{z\}$ = imaginary part of the complex number z. So, the sum of all $n-1$ chords with an endpoint in vertex V_0 is (including the zero term $k = 0$ for convenience):

$$\begin{aligned}
\sum_{k=0}^{n-1} 2\Im \left\{ e^{\frac{\pi k}{n} i} \right\} &= 2\Im \left\{ \sum_{k=0}^{n-1} e^{\frac{\pi k}{n} i} \right\} \\
&= 2\Im \left\{ \frac{e^{\pi i} - 1}{e^{\frac{\pi}{n} i} - 1} \right\} \\
&= 2\Im \left\{ \frac{-2}{e^{\frac{\pi}{n} i} - 1} \right\} \\
&= 2\Im \left\{ \frac{-2e^{-\frac{\pi}{2n} i}}{e^{\frac{\pi}{2n} i} - e^{-\frac{\pi}{2n} i}} \right\} \\
&= 2\Im \left\{ \frac{-2(\cos \frac{\pi}{2n} + i \sin \frac{\pi}{2n})}{2i \sin \frac{\pi}{2n}} \right\} \\
&= \frac{2 \cos \frac{\pi}{2n}}{\sin \frac{\pi}{2n}}\,.
\end{aligned}$$

Repeating the same computation for all chords with an endpoint in V_1, then V_2, and so on until V_{n-1}, we get the same expression n times. Adding and taking into account that each chord has been counted twice, we get that the sum of the lengths of all chords is $n\frac{\cos\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}$. Since there are $\frac{n(n-1)}{2}$ chords, the average is

$$A_n = \frac{2}{n-1}\frac{\cos\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}\,.$$

For the limit we can use L'Hôpital after replacing $n \to \infty$ with $x = \frac{1}{n} \to 0$:

$$\begin{aligned}
\lim_{n\to\infty} A_n &= \lim_{n\to\infty} \frac{2}{n-1}\frac{\cos\frac{\pi}{2n}}{\sin\frac{\pi}{2n}} \\
&= \lim_{x\to\ 0} \frac{2x}{1-x}\frac{\cos\frac{\pi x}{2}}{\sin\frac{\pi x}{2}} \\
&= \lim_{x\to\ 0} \frac{2x}{\sin\frac{\pi x}{2}} \\
&= \lim_{x\to\ 0} \frac{2}{\frac{\pi}{2}\cos\frac{\pi x}{2}} \\
&= \frac{4}{\pi}\,.
\end{aligned}$$