PUTNAM TRAINING
POLYNOMIALS

(Last updated: November 30, 2021)

REMARK. This is a list of exercises on polynomials. —Miguel A. Lerma

EXERCISES

1. Find a polynomial with integral coefficients whose zeros include $\sqrt{2}+\sqrt{5}$.

2. Let $p(x)$ be a polynomial with integer coefficients. Assume that $p(a) = p(b) = p(c) = -1$, where a, b, c are three different integers. Prove that $p(x)$ has no integral zeros.

3. Prove that the sum

$$\sqrt{1001^2+1}+\sqrt{1002^2+1}+\cdots+\sqrt{2000^2+1}$$

is irrational.

4. (USAMO 1975) If $P(x)$ denotes a polynomial of degree n such that $P(k) = k/(k+1)$ for $k = 0, 1, 2, \dots, n$, determine $P(n+1)$.

5. (USAMO 1984) The product of two of the four zeros of the quartic equation

$$x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$$

is -32. Find k.

6. Let n be an even positive integer, and let $p(x)$ be an n-degree polynomial such that $p(-k) = p(k)$ for $k = 1, 2, \dots, n$. Prove that there is a polynomial $q(x)$ such that $p(x) = q(x^2)$.

7. Let $p(x)$ be a polynomial with integer coefficients satisfying that $p(0)$ and $p(1)$ are odd. Show that p has no integer zeros.

8. (USAMO 1976) If $P(x)$, $Q(x)$, $R(x)$, $S(x)$ are polynomials such that

$$P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + x^2 + x + 1)S(x)$$

prove that $x - 1$ is a factor of $P(x)$.

9. Let a, b, c distinct integers. Can the polynomial $(x-a)(x-b)(x-c)-1$ be factored into the product of two polynomials with integer coefficients?

10. Let $p_1, p_2, \dots, p_n$ distinct integers and let $f(x)$ be the polynomial of degree n given by

$$f(x) = (x-p_1)(x-p_2)\cdots(x-p_n)\,.$$

Prove that the polynomial

$$g(x) = (f(x))^2 + 1$$

cannot be expressed as the product of two non-constant polynomials with integral coefficients.

11. Find the remainder when you divide $x^{81} + x^{49} + x^{25} + x^9 + x$ by $x^3 - x$.

12. Does there exist a polynomial $f(x)$ for which $xf(x-1) = (x+1)f(x)$?.

13. Is it possible to write the polynomial $f(x) = x^{105} - 9$ as the product of two polynomials of degree less than 105 with integer coefficients?

14. Find all prime numbers p that can be written $p = x^4 + 4y^4$, where x, y are positive integers.

15. (Canada, 1970) Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with integral coefficients. Suppose that there exist four distinct integers a, b, c, d with $P(a) = P(b) = P(c) = P(d) = 5$. Prove that there is no integer k with $P(k) = 8$.

16. Show that $(1 + x + \cdots + x^n)^2 - x^n$ is the product of two polynomials.

17. Let $f(x)$ be a polynomial with real coefficients, and suppose that $f(x) + f'(x) > 0$ for all x. Prove that $f(x) > 0$ for all x.

18. Evaluate the following determinant: $\begin{vmatrix} 1 & 1 & 1 & 1 \\ w & x & y & z \\ w^2 & x^2 & y^2 & z^2 \\ w^3 & x^3 & y^3 & z^3 \end{vmatrix}$

19. Evaluate the following determinant: $\begin{vmatrix} 1 & 1 & 1 & 1 \\ w & x & y & z \\ w^2 & x^2 & y^2 & z^2 \\ w^4 & x^4 & y^4 & z^4 \end{vmatrix}$

20. Do there exist polynomials a, b, c, d such that $1 + xy + x^2y^2 = a(x)b(y) + c(x)d(y)$?

21. Determine all polynomials such that $P(0) = 0$ and $P(x^2 + 1) = P(x)^2 + 1$.

22. Consider the lines that meet the graph

$$y = 2x^4 + 7x^3 + 3x - 5$$

in four distinct points $P_i = [x_i, y_i]$, $i = 1, 2, 3, 4$. Prove that

$$\frac{x_1 + x_2 + x_3 + x_4}{4}$$

is independent of the line, and compute its value.

23. Let k be the smallest positive integer for which there exist distinct integers a, b, c, d, e such that

$$(x-a)(x-b)(x-c)(x-d)(x-e)$$

has exactly k nonzero coefficients. Find, with proof, a set of integers for which this minimum k is achieved.

24. Find the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.

25. (Putnam 1999, A1) Find polynomials $f(x)$, $g(x)$, and $h(x)$ such that

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1\,, & \text{if } x < -1\,, \\ 3x+2\,, & \text{if } -1 \leq x \leq 0\,, \\ -2x+2\,, & \text{if } x > 0\,. \end{cases}$$

26. Suppose that α, β, and γ are real numbers such that

$$\alpha+\beta+\gamma = 2\,,$$
$$\alpha^2+\beta^2+\gamma^2 = 14\,,$$
$$\alpha^3+\beta^3+\gamma^3 = 17\,.$$

Find $\alpha\beta\gamma$.

27. Prove that $(2+\sqrt{5})^{1/3} - (-2+\sqrt{5})^{1/3}$ is rational.

28. Two players A and B play the following game. A thinks of a polynomial with non-negative integer coefficients. B must guess the polynomial. B has two shots: she can pick a number and ask A to return the polynomial value there, and then she has another such try. Can B win the game?

29. Let $f(x)$ a polynomial with real coefficients, and suppose that $f(x)+f'(x) > 0$ for all x. Prove that $f(x) > 0$ for all x.

30. If $a, b, c > 0$, is it possible that each of the polynomials $P(x) = ax^2+bx+c$, $Q(x) = cx^2+ax+b$, $R(x) = bx^2+cx+a$ has two real roots?

31. Let $f(x)$ and g(x) be nonzero polynomials with real coefficients such that $f(x^2+x+1) = f(x)g(x)$. Show that $f(x)$ has even degree.

32. Prove that there is no polynomial $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ with integer coefficients and of degree at least 1 with the property that $P(0)$, $P(1)$, $P(2)$,..., are all prime numbers.

33. Prove that if m is a positive integer not a multiple of 3 then the polynomial $p(x) = x^{2m}+x^m+1$ is a multiple of $q(x) = x^2+x+1$, i.e., there is a polynomial with real coefficients $r(x)$ such that $p(x) = q(x)r(x)$.

Hints

1. Call $x = \sqrt{2} + \sqrt{5}$ and eliminate the radicals.

2. Factor $p(x) + 1$.

3. Prove that the sum is the root of a monic polynomial but not an integer.

4. Look at the polynomial $Q(x) = (x + 1)P(x) - x$.

5. Use the relationship between zeros and coefficients of a polynomial.

6. The $(n - 1)$-degree polynomial $p(x) - p(-x)$ vanishes at n different points.

7. For each integer k study the parity of $p(k)$ depending on the parity of k.

8. We must prove that $P(1) = 0$. See what happens by replacing x with fifth roots of unity.

9. Assume $(x - a)(x - b)(x - c) - 1 = p(x)q(x)$, and look at the possible values of $p(x)$ and $q(x)$ for $x = a, b, c$.

10. Assume $g(x) = h(x)k(x)$, where $h(x)$ and $k(x)$ are non-constant polynomials with integral coefficients. Prove that the can be assumed to be positive for every x and $h(p_i) = k(p_i) = 1$, $i = 1, \dots, n$. Deduce that both are of degree n and determine their form. Get a contradiction by equating coefficients in $g(x)$ and $h(x)k(x)$.

11. The remainder will be a second degree polynomial. Plug the roots of $x^3 - x$.

12. Find the value of $f(n)$ for n integer.

13. Assume $f(x) = g(x)h(x)$, where $g(x)$ and $h(x)$ have integral coefficients and degree less than 105. Look at the product of the roots of $g(x)$

14. Sophie Germain's Identity.

15. We have that a, b, c, d are distinct roots of $P(x) - 5$.

16. One way to solve this problem is by letting $A_{n-1} = 1 + x + \cdots + x^{n-1}$ and doing some algebra.

17. Study the behavior of $f(x)$ as $x \to \pm\infty$. Also determine the number of roots of $f(x)$.

18. Expand the determinant along the last column and find its zeros as a polynomial in z.

19. Expand the determinant along the last column and find its zeros as a polynomial in z.

20. Write the given condition in matrix form and give each of x and y three different values.

21. Find some polynomial that coincides with $P(x)$ for infinitely many values of x.

22. Find intersection points solving a system of equations.

23. The numbers a, b, c, d, e are the roots of the given polynomial. How are the roots of a fifth-degree polynomial with exactly 1,2,... non-zero coefficients?

24. Find first the set of x verifying the constrain.

25. Try with first degree polynomials. Some of those polynomials must change sign precisely at $x = -1$ and $x = 0$. Recall that $|u| = \pm u$ depending on whether $u \geq 0$ or $u < 0$.

26. Write the given sums of powers as functions of the elementary symmetric polynomials of α, β, γ.

27. Find a polynomial with integer coefficients with that number as one of its roots.

28. What happens if B has an upper bound for the coefficients of the polynomials?

29. How could $f(x)$ become zero, and how many times? From the behavior of $f(x)+f'(x)$, what can we conclude about the leading coefficient and degree of $f(x)$?

30. What conditions must the coefficients satisfy for a second degree polynomial to have two real roots?

31. Prove that f(x) cannot have real roots.

32. We have that $a_0 = P(0)$ must be a prime number.

33. Show that the roots of $x^2 + x + 1$ are also roots of $x^{2m} + x^m + 1$.

Solutions

1. If $x = \sqrt{2} + \sqrt{5}$ then

$$x^2 = 7 + 2\sqrt{10}\,,$$
$$x^2 - 7 = 2\sqrt{10}\,,$$
$$(x^2 - 7)^2 = 40\,,$$
$$x^4 - 14x^2 + 9 = 0\,.$$

Hence the desired polynomial is $x^4 - 14x^2 + 9$.

2. We have that $p(x)+1$ has zeros at a, b, and c, hence $p(x)+1 = (x-a)(x-b)(x-c)q(x)$. If p had an integral zero d we would have

$$(d-a)(d-b)(d-c)q(d) = 1\,,$$

where $d-a$, $d-b$, and $d-c$ are distinct integers. But that is impossible, because 1 has only two possible factors, 1 and -1.

3. We prove it by showing that the sum is the root of a monic polynomial but not an integer—so by the rational roots theorem it must be irrational.
First we notice that $n < \sqrt{n^2+1} < n + 1/n$, hence the given sum is of the form

$$S = 1001 + \theta_1 + 1002 + \theta_2 + \cdots + 2000 + \theta_{1000}$$

were $0 < \theta_i < 1/1001$, consequently

$$0 < \theta_1 + \theta_2 + \cdots + \theta_{1000} < 1\,,$$

so S is not an integer.
Now we must prove that S is the root of a monic polynomial. More generally we will prove that a sum of the form

$$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n}$$

where the a_i's are positive integers, is the root of a monic polynomial.[1] This can be proved by induction on n. For $n = 1$, $\sqrt{a_1}$ is the root of the monic polynomial $x^2 - a_1$. Next assume that $y = \sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n}$ is a zero of a monic polynomial $P(x) = x^r + c_{r-1}x^{r-1} + \cdots + c_0$. We will find a polynomial that has $z = y + \sqrt{a_{n+1}}$ as a zero. We have

$$0 = P(y) = P(z - \sqrt{a_{n+1}}) = (z - \sqrt{a_{n+1}})^r + c_{r-1}(z - \sqrt{a_{n+1}})^{r-1} + \cdots + c_0\,.$$

Expanding the parentheses and grouping the terms that contain $\sqrt{a_{n+1}}$:

$$0 = P(z - \sqrt{a_{n+1}}) = z^r + Q(z) + \sqrt{a_{n+1}}\,R(z)\,.$$

[1]The result can be obtained also by resorting to a known theorem on *algebraic integers* ("algebraic integer" is the mathematical term used to designate a root of a monic polynomial.) It is known that algebraic integers form a mathematical structure called *ring*, basically meaning that the sum, difference and product of two algebraic integers is an algebraic integer. Now, if a_i and k_i are positive integers, then $\sqrt[k_i]{a_i}$ is an algebraic integer, because it is a root of the monic polynomial $x^{k_i} - a_i$. Next, since the sum or difference of algebraic integers is an algebraic integer then $\pm\sqrt[k_1]{a_1} \pm \sqrt[k_2]{a_2} \pm \cdots \pm \sqrt[k_n]{a_n}$ is in fact an algebraic integer (note that the roots do not need to be square roots, and the signs can be combined in any way.)

Putting radicals on one side and squaring

$$(z^r + Q(z))^2 = a_{n+1}\left(R(z)\right)^2 ,$$

so

$$T(x) = (x^r + Q(x))^2 - a_{n+1}\left(R(x)\right)^2$$

is a monic polynomial with z as a root.

4. Consider the following polynomial:

$$Q(x) = (x+1)P(x) - x \,.$$

We have that $Q(k) = 0$ for $k = 0, 1, 2, \dots, n$, hence, by the *Factor theorem*,

$$Q(x) = Cx(x-1)(x-2)\dots(x-n) \,,$$

where C is a constant to be determined. Plugging $x = -1$ we get

$$Q(-1) = C(-1)(-2)\cdots(-(n+1)) \,.$$

On the other hand $Q(-1) = 0 \cdot P(-1) - (-1) = 1$, hence $C = \dfrac{(-1)^{n+1}}{(n+1)!}$.
Next, plugging in $x = n+1$ we get

$$(n+2)P(n+1) - (n+1) = C(n+1)! = \frac{(-1)^{n+1}}{(n+1)!}(n+1)! = (-1)^{n+1} \,,$$

hence

$$\boxed{P(n+1) = \frac{n+1+(-1)^{n+1}}{n+2}} \,.$$

5. Let the zeros be a, b, c, d. The relationship between zeros and coefficients yields

$$\begin{aligned} a+b+c+d &= 18 \\ ab+ac+ad+bc+bd+cd &= k \\ abc+abd+acd+bcd &= -200 \\ abcd &= -1984 \,. \end{aligned}$$

Assume $ab = -32$ and let $u = a+b$, $v = c+d$, $w = cd$. Then

$$\begin{aligned} u+v &= 18 \\ -32+uv+w &= k \\ -32v+uw &= -200 \\ -32w &= -1984 \,. \end{aligned}$$

From the last equation we get $w = 62$, and replacing in the other equations we easily get $u = 4$, $v = 14$. Hence

$$k = -32 + 4 \cdot 14 + 62 = \boxed{86} \,.$$

6. Let $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$. The $(n-1)$-degree polynomial $p(x) - p(-x) = 2(a_1 x + a_3 x^3 + \cdots + a_{n-1} x^{n-1})$ vanishes at n different points, hence, it must be identically null, i.e., $a_1 = a_3 = \cdots = a_{n-1} = 0$. Hence $p(x) = a_0 + a_2 x^2 + a_4 x^4 \cdots + a_n x^n$, and $q(x) = a_0 + a_2 x + a_4 x^2 \cdots + a_n x^{n/2}$.

7. If k is an even integer we have $p(k) \equiv p(0) \pmod 2$, and if it is odd then $p(k) \equiv p(1) \pmod 2$. Since $p(0)$ and $p(1)$ are odd we have $p(k) \equiv 1 \pmod 2$ for every integer k, so $p(k)$ cannot be zero.

8. We must prove that $P(1) = 0$. Consider the four complex numbers $\rho_k = e^{2\pi i k/5}$, $k = 1, 2, 3, 4$. All of them verify $\rho_k^5 = 1$, so together with 1 they are the roots of $x^5 - 1$. Since $x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$ then the ρ_k's are the roots of $x^4 + x^3 + x^2 + x + 1$. So

$$P(1) + \rho_k Q(1) + \rho_k^2 R(1) = 0\,.$$

Adding for $k = 1, 2, 3, 4$ and taking into account that the numbers ρ_k^2 are the ρ_k's in a different order we get

$$P(1) = 0\,.$$

9. The answer is no. We prove it by contradiction. Assume $(x-a)(x-b)(x-c) - 1 = p(x)q(x)$, where p is linear and q is quadratic. Then $p(a)q(a) = p(b)q(b) = p(c)q(c) = -1$. If the coefficients of p and q must integers they can take only integer values, so in each product one of the factor must be 1 and the other one is -1. Hence either $p(x)$ takes the value 1 twice or it takes the value -1 twice. But a 1st degree polynomial cannot take the same value twice.

10. We prove it by contradiction. Suppose $g(x) = h(x)k(x)$, where $h(x)$ and $k(x)$ are non-constant polynomials with integral coefficients. Since $g(x) > 0$ for every x, $h(x)$ and $k(x)$ cannot have real roots, so they cannot change signs and we may suppose $h(x) > 0$ and $k(x) > 0$ for every x. Since $g(p_i) = 1$ for $i = 1, \dots, n$, we have $h(p_i) = k(p_i) = 1$, $i = 1, \dots, n$. If either $h(x)$ or $k(x)$ had degree less than n, it would constant, against the hypothesis, so they must be of degree n. Also we know that $h(x) - 1$ and $k(x) - 1$ are zero for $x = p_i$, $i = 1, \dots, n$, so their roots are precisely $p_1, \dots, p_n$, and we can write

$$\begin{aligned} h(x) &= 1 + a(x - p_1) \cdots (x - p_n) \\ k(x) &= 1 + b(x - p_1) \cdots (x - p_n)\,, \end{aligned}$$

where a and b are integers. So we have

$$(x - p_1)^2 \cdots (x - p_n)^2 + 1 = \\ 1 + (a+b)(x - p_1) \cdots (x - p_n) + ab(x - p_1)^2 \cdots (x - p_n)^2\,.$$

Hence

$$\begin{cases} a + b = 0 \\ ab = 1\,, \end{cases}$$

which is impossible, because there are no integers a, b verifying those equations.

11. Assume the quotient is $q(x)$ and the remainder is $r(x) = ax^2 + bx + c$. Then

$$x^{81} + x^{49} + x^{25} + x^9 + x = q(x)(x^3 - x) + r(x)\,,$$

Plugging in the values $x = -1, 0, 1$ we get $r(-1) = -5$, $r(0) = 0$, $r(1) = 5$. From here we get $a = c = 0$, $b = 5$, hence the remainder is $r(x) = 5x$.

12. For positive integer n we have $f(n) = \frac{n}{n+1} f(n-1) = \frac{n-1}{n+1} f(n-2) = \cdots = 0 \cdot f(-1) = 0$. Hence $f(x)$ has infinitely many zeros, and must be identically zero $f(x) \equiv 0$.

13. By contradiction. Assume $f(x) = g(x)h(x)$, where $g(x)$ and $h(x)$ have integral coefficients and degree less than 105. Let $\alpha_1, \dots, \alpha_k$ the (complex) roots of $g(x)$. For each $j = 1, \dots, k$ we have $\alpha_j^{105} = 9$, hence $|\alpha_j| = \sqrt[105]{9}$, and $|\alpha_1 \alpha_2 \cdots \alpha_k| = (\sqrt[105]{9})^k =$ the absolute value of the constant term of $g(x)$ (an integer.) But $(\sqrt[105]{9})^k = \sqrt[105]{3^{2k}}$ cannot be an integer.

14. The answer is $p = 5$. By Sophie Germain's Identity we have

$$x^4 + 4y^4 = (x^2 + 2y^2 + 2xy)(x^2 + 2y^2 - 2xy) = [(x+y)^2 + y^2][(x-y)^2 + y^2]\,,$$

which can be prime only if $x = y = 1$.

15. We have that a, b, c, d are distinct roots of $P(x) - 5$, hence

$$P(x) - 5 = g(x)(x-a)(x-b)(x-c)(x-d)\,,$$

where $g(x)$ is a polynomial with integral coefficients. If $P(k) = 8$ then

$$g(x)(x-a)(x-b)(x-c)(x-d) = 3\,,$$

but 3 is a prime number, so all the factors on the left but one must be ± 1. So among the numbers $(x-a)$, $(x-b)$, $(x-c)$, $(x-b)$, there are either two 1's or two -1's, which implies that a, b, c, d cannot be all distinct, a contradiction.

16. Calling $A_{n-1} = 1 + x + \cdots + x^{n-1}$, we have

$$\begin{aligned}
(1 + x + \cdots + x^n)^2 - x^n &= (A_{n-1} + x^n)^2 - x^n \\
&= A_{n-1}^2 + 2A_{n-1}x^n + x^{2n} - x^n \\
&= A_{n-1}^2 + 2A_{n-1}x^n + (x^n - 1)x^n \\
&= A_{n-1}^2 + 2A_{n-1}x^n + A_{n-1}(x-1)x^n \\
&= A_{n-1}(A_{n-1} + 2x^n + (x-1)x^n) \\
&= A_{n-1}(A_{n-1} + x^n + x^{n+1}) \\
&= (1 + x + \cdots + x^{n-1})(1 + x + \cdots + x^{n+1})\,.
\end{aligned}$$

17. Since $f(x)$ and $f(x) + f'(x)$ have the same leading coefficient, the limit of $f(x)$ as $x \to \pm\infty$ must be equal to that of $f(x) + f'(x)$, i.e., $+\infty$.
Note that f cannot have multiple real roots, because at any of those roots both f and f' would vanish, contradicting the hypothesis. So all real roots of f, if any, must be simple roots.

Since $f(x) \to +\infty$ for both $x \to \infty$ and $x \to -\infty$, it must have an even number of real roots (if any): $x_1 < x_2 < \cdots < x_{2n}$. Note that between x_1 and x_2, $f(x)$ must be negative, and by Rolle's theorem its derivative must be zero at some intermediate point $a \in (x_1, x_2)$, hence $f(a) + f'(a) = f(a) < 0$, again contradicting the hypothesis. Consequently, $f(x)$ has no real roots, and does not change sign at any point, which implies $f(x) > 0$ for all x.

18. This is a particular case of the well known Vandermonde determinant, but here we will find its value using arguments from polynomial theory. Expanding the determinant along the last column using Laplace formula we get

$$a_0(w,x,y) + a_1(w,x,y)z + a_2(w,x,y)z^2 + a_3(w,x,y)z^3\,,$$

where $a_i(w,x,y)$ is the cofactor of z^i.
Since the determinant vanishes when two columns are equal, that polynomial in z has zeros at $z = w$, $z = x$, $z = y$, hence it must be of the form

$$a_3(w,x,y,z)(z-y)(z-x)(z-w)\,.$$

Note that $a_3(w,x,y) = \begin{vmatrix} 1 & 1 & 1 \\ w & x & y \\ w^2 & x^2 & y^2 \end{vmatrix}$, which can be computed in an analogous way:

$$a_3(w,x,y) = b_2(w,x)(y-x)(y-w)\,,$$
$$b_2(w,x) = c_1(w)(x-w)\,,$$
$$c_1(w) = 1\,.$$

Hence the value of the given determinant is

$$(z-y)(z-x)(z-w)(y-x)(y-w)(x-w)\,.$$

19. Expanding the determinant along the last column using Laplace formula we get

$$a_0(w,x,y) + a_1(w,x,y)z + a_2(w,x,y)z^2 + a_4(w,x,y)z^4\,,$$

where $a_i(w,x,y)$ is the cofactor of z^i. In particular $a_4(w,x,y) = (y-x)(y-w)(x-w)$ by Vandermonde formula.
Since the determinant vanishes when two columns are equal, that polynomial in z has zeros at $z = w$, $z = x$, $z = y$, hence it must be of the form

$$a_4(w,x,y,z)(z-y)(z-x)(z-w)\,b(w,x,y,z) =$$
$$(z-y)(z-x)(z-w)(y-x)(y-w)(x-w)\,b(w,x,y,z)\,,$$

where $b(w,x,y,z)$ is some first degree homogeneous polynomial in w,x,y,z. Note that the value of $b(w,x,y,z)$ won't change by permutations of its arguments, so $b(w,x,y,z)$ is symmetric, and all its coefficients must be equal, hence $b(w,x,y,z) = k\cdot(w+x+y+z)$ for some constant k. The value of k can be found by computing the determinant for particular values of w,x,y,z, say $w = 0$, $x = 1$, $y = 2$, $z = 3$, and we obtain $k = 1$. Hence the value of the determinant is

$$(z-y)(z-x)(z-w)(y-x)(y-w)(x-w)(w+x+y+z)\,.$$

20. The answer is no.
We can write the condition in matrix form in the following way:

$$\begin{pmatrix} 1 & x & x^2 \end{pmatrix} \begin{pmatrix} 1 \\ y \\ y^2 \end{pmatrix} = \begin{pmatrix} a(x) & c(x) \end{pmatrix} \begin{pmatrix} b(y) \\ d(y) \end{pmatrix} .$$

By assigning values $x = 0, 1, 2$, $y = 0, 1, 2$, we obtain the following identity:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} a(0) & c(0) \\ a(1) & c(1) \\ a(2) & c(2) \end{pmatrix} \begin{pmatrix} b(0) & b(1) & b(2) \\ d(0) & d(1) & d(2) \end{pmatrix} .$$

The product on the left hand side yields a matrix of rank 3, while the right hand side has rank at most 2, contradiction.

21. The answer is $P(x) = x$.
In order to prove this we show that $P(x)$ equals x for infinitely many values of x. In fact, let a_n the sequence $0, 1, 2, 5, 26, 677, \dots$, defined recursively $a_0 = 0$, and $a_{n+1} = a_n^2 + 1$ for $n \geq 0$. We prove by induction that $P(a_n) = a_n$ for every $n = 0, 1, 2, \dots$. In the basis case, $n = 0$, we have $P(0) = 0$. For the induction step assume $n \geq 1$, $P(a_n) = a_n$. Then $P(a_{n+1}) = P(a_n^2 + 1) = P(a_n)^2 + 1 = a_n^2 + 1 = a_{n+1}$.
Since in fact $P(x)$ coincides with x for infinitely many values of x, we must have $P(x) = x$ identically.

22. Given a line $y = mx + b$, the intersection points with the given curve can be computed by solving the following system of equations

$$\begin{cases} y = 2x^4 + 7x^3 + 3x - 5 \\ y = mx + b \, . \end{cases}$$

Subtracting we get $2x^4 + 7x^3 + 3(x - m) - 5 - b = 0$. If the line intersects the curve in four different points, that polynomial will have four distinct roots x_1, x_2, x_3, x_4, and their sum will be minus the coefficient of x^3 divided by the coefficient of x^4, i.e., $-7/2$, hence

$$\frac{x_1 + x_2 + x_3 + x_4}{4} = -\frac{7}{8} \, .$$

23. The answer is $k = 3$, and an example is

$$(x + 2)(x + 1)x(x - 1)(x - 2) = x^5 - 5x^3 + 4x \, ,$$

where $\{-2, -1, 0, 1, 2\}$ is the desired set of integers.
To complete the argument we must prove that k cannot be less than 3. It cannot be 1 because in that case the polynomial would be x^5, with all its five roots equal zero (note that a, b, c, d, e are the roots of the polynomial, and by hypothesis they must be distinct integers). Assume now that $k = 2$. Then the polynomial would be of the form $x^5 + nx^i = x^i(x^{5-i} + n)$, n a nonzero integer, $0 \leq i \leq 4$. If $i \geq 2$ then the polynomial would have two or more roots equal zero, contradicting the hypothesis. If $i = 1$ then the roots of the polynomial would be 0, and the roots of $x^4 + n$, at least two of which

are non-real complex roots. If $i = 0$ then the polynomial is $x^5 + n$, which has four non-real complex roots.

24. The zeros of $x^4 - 13x^2 + 36$ are $x = \pm 2$ and ± 3, hence the condition is equivalent to $x \in [-3, -2] \cup [2, 3]$. On the other hand $f'(x) = 3x^2 - 3$, with zeros at $x = \pm 1$. This implies that $f(x)$ is monotonic on $[-3, -2]$ and $[2, 3]$, hence (with the given constrain) its maximum value can be attained only at the boundaries of those intervals. Computing $f(-3) = -18$, $f(-2) = -2$, $f(2) = 2$, $f(3) = 18$, we get that the desired maximum is 18.

25. Note that if $r(x)$ and $s(x)$ are any two functions, then

$$\max(r, s) = (r + s + |r - s|)/2.$$

Therefore, if $F(x)$ is the given function, we have

$$\begin{aligned} F(x) &= \max\{-3x - 3, 0\} - \max\{5x, 0\} + 3x + 2 \\ &= (-3x - 3 + |3x - 3|)/2 \\ &\quad - (5x + |5x|)/2 + 3x + 2 \\ &= |(3x - 3)/2| - |5x/2| - x + \frac{1}{2}, \end{aligned}$$

so we may set $f(x) = (3x - 3)/2$, $g(x) = 5x/2$, and $h(x) = -x + \frac{1}{2}$.

26. Writing the given sums of powers as functions of the elementary symmetric polynomials of α, β, γ, we have

$$\begin{aligned} \alpha + \beta + \gamma &= s\,, \\ \alpha^2 + \beta^2 + \gamma^2 &= s^2 - 2q\,, \\ \alpha^3 + \beta^3 + \gamma^3 &= s^3 - 3qs + 3p\,, \end{aligned}$$

where $s = \alpha + \beta + \gamma$, $q = \alpha\beta + \beta\gamma + \alpha\gamma$, $p = \alpha\beta\gamma$.
So we have $s = 2$, and from the second given equation get $q = -5$. Finally from the third equation we get $p = -7$. So, this is the answer, $\alpha\beta\gamma = -7$.
(Note: this is not needed to solve the problem, but by solving the equation $x^3 - sx^2 + qs - p = x^3 - 2x^2 + 5x + 7 = 0$ we find that the three numbers α, β, and γ are approx. 2.891954442, -2.064434534, and 1.172480094.)

27. Let $\alpha = (2 + \sqrt{5})^{1/3} - (-2 + \sqrt{5})^{1/3}$. By raising to the third power, expanding and simplifying we get that α verifies the following polynomial equation:

$$\alpha^3 + 3\alpha - 4 = 0\,.$$

We have $x^3 + 3x - 4 = (x - 1)(x^2 + x + 4)$. The second factor has no real roots, hence $x^3 + 3x - 4$ has only one real root equal to 1, i.e., $\alpha = 1$.

28. The answer is affirmative, B can in fact guess the polynomial—call it $f(x) = a_0 + a_1x^2 + a_2x^2 + \cdots a_nx^n$. By asking A to evaluate it at 1, B gets an upper bound $f(1) = a_0 + a_1 + a_2 + \cdots a_n = M$ for the coefficients of the polynomial. Then, for any integer $N > M$, the coefficients of the polynomial are just the digits of $f(N) = a_0 + a_1N^2 + a_2N^2 + \cdots a_nN^n$ in base N.

29. If $f(x_0) = 0$ at some point x_0, then by hypothesis we would have $f'(x_0) > 0$, and f would be (strictly) increasing at x_0. This implies:
(1) If the polynomial f becomes zero at some point x_0, then $f(x) > 0$ for every $x > x_0$, and $f(x) < 0$ for every $x < x_0$.
Writing $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$, we have $f(x) + f'(x) = a_nx^n + (a_{n-1} + na_n)x^{n-1} + \cdots + (a_0 + a_1)$. Given that that $f(x) + f'(x) > 0$ for all x, we deduce that $a_n = \lim_{n\to\infty} \frac{f(x)+f'(x)}{x^n} > 0$. On the other hand n must be even, otherwise $f(x)+f'(x)$ would become negative as $x \to -\infty$. Hence $f(x) > 0$ for $|x|$ large enough. By (1) this rules out the possibility of $f(x)$ becoming zero at some point x_0, and so it must be always positive.
Remark: The statement is not true for functions in general, e.g., $f(x) = -e^{-2x}$ verifies $f(x) + f'(x) = e^{-2x} > 0$, but $f(x) < 0$ for every x.

30. The answer is No. If $P(x)$ has two real roots we would have $b^2 > 4ac$. Analogously for $R(x)$ and $Q(x)$ we should have $a^2 > 4cb$, and $c^2 > 4ab$ respectively. Multiplying the inequalities we get $a^2b^2c^2 > 64a^2b^2c^2$, which is impossible.

31. First we prove (by contradiction) that $f(x)$ has no real roots. In fact, if x_1 is a real root of $f(x)$, then we have that $x_2 = x_1^2 + x_1 + 1$ is also a real root of $f(x)$, because $f(x_1^2+x_1+1) = f(x_1)g(x_1) = 0$. But $x_1^2+1 > 0$, hence $x_2 = x_1^2+x_1+1 > x_1$. Repeating the reasoning we get that $x_3 = x_2^2+x_2+1$ is another root of $f(x)$ greater than x_2, and so on, so we get an infinite increasing sequence of roots of $f(x)$, which is impossible. Consequently $f(x)$ must have even degree, because all odd degree polynomials with real coefficients have at least one real root. Q.E.D.
Note: An example of a polynomial with the desired property is: $f(x) = x^2 + 1$, $f(x^2 + x + 1) = (x^2 + 1)(x^2 + 2x + 2)$.
Remark: The result is not generally true for polynomials with complex coefficients—counterexample: $f(x) = x + i$, $f(x^2 + x + 1) = x^2 + x + 1 + i = (x + i)(x + 1 - i)$.

32. By contradiction. We have that $a_0 = P(0)$ must be a prime number. Also, $P(ka_0)$ is a multiple of a_0 for every $k = 0, 1, 2, \dots$, but if $P(ka_0)$ is prime then $P(ka_0) = a_0$ for every $k \geq 0$. This implies that the polynomial $Q(x) = P(a_0x) - a_0$ has infinitely many roots, so it is identically zero, and $P(a_0x) = a_0$, contradicting the hypothesis that P is of degree at least 1.

33. We have $q(x) = x^2 + x + 1 = (x - \omega)(x - \omega^2)$, where $\omega = e^{2\pi/3}$. Note that $(\pm\omega)^3 = 1$. Also note that since m is not a multiple of 3, then $m = 3k \pm 1$, hence

$$p(\omega) = \omega^{6k\pm2} + \omega^{3k\pm1} + 1 = \omega^{\pm2} + \omega^{\pm1} + 1 = \omega^2 + \omega + 1 = q(\omega) = 0\,,$$

where we have used $\omega^{-2} = \omega$, and $\omega^{-1} = \omega^2$. So all the roots of $q(x)$ are roots of $p(x)$, and this implies that in fact $p(x)$ is a multiple of $q(x)$.