MATHEMATICAL INDUCTION

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Mathematical Induction

This is a powerful method to prove properties of positive integers.

1. **Principle of Mathematical Induction.** Let P be a property of positive integers such that:

(1) *Base Case*: $P(1)$ is true, and

(2) *Inductive Step*: if $P(n)$ is true, then $P(n+1)$ is true.

Then $P(n)$ is true for all positive integers.

The premise $P(n)$ in the inductive step is called *Induction Hypothesis*.

The validity of the Principle of Mathematical Induction is obvious. The base case states that $P(1)$ is true. Then the inductive step implies that $P(2)$ is also true. By the inductive step again we see that $P(3)$ is true, and so on. Consequently the property must be true for all positive integers:

$$\overset{\substack{\text{base case}\\ \downarrow}}{\underset{\text{true}}{P(1)}} \underset{\substack{\text{inductive}\\ \text{step}}}{\longrightarrow} \underset{\text{true}}{P(2)} \underset{\substack{\text{inductive}\\ \text{step}}}{\longrightarrow} \underset{\text{true}}{P(3)} \underset{\substack{\text{inductive}\\ \text{step}}}{\longrightarrow} \cdots$$

$$\cdots \underset{\substack{\text{inductive}\\ \text{step}}}{\longrightarrow} \underset{\text{true}}{P(n)} \underset{\substack{\text{inductive}\\ \text{step}}}{\longrightarrow} \underset{\text{true}}{P(n+1)} \underset{\substack{\text{inductive}\\ \text{step}}}{\longrightarrow} \cdots$$

Example: Prove that the sum of the n first odd positive integers is n^2, i.e., $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

Answer: Let $S(n) = 1 + 3 + 5 + \cdots + (2n - 1)$. We want to prove by induction that for every positive integer n, $S(n) = n^2$.

(1) *Base Case*: If $n = 1$ we have $S(1) = 1 = 1^2$, so the property is true for 1.

(2) *Inductive Step*: Assume (*Induction Hypothesis*) that the property is true for some positive integer n, i.e.: $S(n) = n^2$. We must prove that it is also true for $n+1$, i.e., $S(n+1) = (n+1)^2$. In fact:

$$S(n+1) = \underbrace{1 + 3 + 5 + \cdots + (2n-1)}_{S(n)} + (2n+1) = S(n) + 2n + 1\,.$$

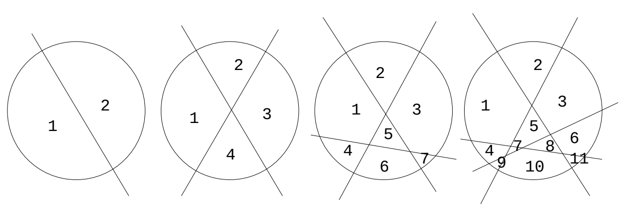
But by induction hypothesis, $S(n) = n^2$, hence:

$$S(n+1) = n^2 + 2n + 1 = (n+1)^2\,.$$

This completes the induction, and shows that the property is true for all positive integers. □

Example: Find the number $R(n)$ of regions in which the plane can be divided by n straight lines.

Answer: By experimentation we easily find:



n	1	2	3	4	...
$R(n)$	2	4	7	11	...

A formula that fits the first few cases is $R(n) = (n^2 + n + 2)/2$, but we need to make sure whether it works for all $n \geq 1$. So we use mathematical induction.

(1) *Base Case*: For $n = 1$ we have $R(1) = 2 = (1^1 + 1 + 2)/2$, which is correct.

(2) *Inductive Step*: Assume (*Induction Hypothesis*) that the property is true for some positive integer n, i.e.:

$$R(n) = \frac{n^2 + n + 2}{2}\,.$$

We must prove that it is also true for $n+1$, i.e.,

$$R(n+1) = \frac{(n+1)^2 + (n+1) + 2}{2} = \frac{n^2 + 3n + 4}{2}\,.$$

So lets look at what happens when we introduce the $n+1$-th straight line. In general this line will intersect the other n lines in n different intersection points, and it will be divided into $n+1$ segments by those intersection points. Each of those $n+1$ segments divides a previous region into two regions, so the number of regions increases by $n+1$. Hence:

$$R(n+1) = S(n) + n + 1\,.$$

But by induction hypothesis, $R(n) = (n^2 + n + 2)/2$, hence:

$$R(n+1) = \frac{n^2 + n + 2}{2} + n + 1 = \frac{n^2 + 3n + 4}{2}\,.$$

Q.E.D.

This completes the induction, and shows that the property is true for all positive integers. □

2. **Generalization of the Base Case.** In the base case we may replace 1 with some other integer m, and then the conclusion is that the property is true for every integer n greater than or equal to m.

Example: Prove that $2n + 1 \leq 2^n$ for $n \geq 3$.

Answer: This is an example in which the property is not true for all positive integers but only for integers greater than or equal to 3.

(1) *Base Case*: If $n = 3$ we have $2n + 1 = 2 \cdot 3 + 1 = 7$ and $2^n = 2^3 = 8$, so the property is true in this case.

(2) *Inductive Step*: Assume (*Induction Hypothesis*) that the property is true for some positive integer n, i.e.: $2n + 1 \leq 2^n$. We must prove that it is also true for $n+1$, i.e., $2(n+1)+1 \leq 2^{n+1}$. By the induction hypothesis we know that $2n \leq 2^n$, and we also have that $3 \leq 2^n$ if $n \geq 3$, hence

$$2(n+1) + 1 = 2n + 3 \leq 2^n + 2^n = 2^{n+1}\,.$$

This completes the induction, and shows that the property is true for all $n \geq 3$. □

3. **Strong Form of Mathematical Induction.** In some cases assuming just $P(n)$ is not enough to prove $P(n+1)$, we need to use the fact that the property is true for all cases from the base case to the given n. This strengthening of the induction hypothesis produces the strong form of mathematical induction:

Let P be a property of positive integers such that:

(1) *Base Case*: $P(1)$ is true, and

(2) *Inductive Step*: if $P(k)$ is true for all $1 \le k \le n$ then $P(n+1)$ is true.

Then $P(n)$ is true for all positive integers.

Example: Prove that every integer $n \ge 2$ is prime or a product of primes. *Answer*:

(1) *Base Case*: 2 is a prime number, so the property holds for $n = 2$.

(2) *Inductive Step*: Assume that if $2 \le k \le n$, then k is a prime number or a product of primes. Now, either $n+1$ is a prime number or it is not. If it is a prime number then it verifies the property. If it is not a prime number, then it can be written as the product of two positive integers, $n+1 = k_1\, k_2$, such that $1 < k_1, k_2 < n+1$. By induction hypothesis each of k_1 and k_2 must be a prime or a product of primes, hence $n+1$ is a product of primes.

This completes the proof. □

4. **The Well-Ordering Principle.** Every nonempty set of positive integers has a smallest element.

Example: Prove that $\sqrt{2}$ is irrational (i.e., $\sqrt{2}$ cannot be written as a quotient of two positive integers) using the well-ordering principle.

Answer: Assume that $\sqrt{2}$ is rational, i.e., $\sqrt{2} = a/b$, where a and b are integers. Note that since $\sqrt{2} > 1$ then $a > b$. Now we have $2 = a^2/b^2$, hence $2\,b^2 = a^2$. Since the left hand side is even, then a^2 is even, but this implies that a itself is even, so $a = 2\,a'$. Hence: $2\,b^2 = 4\,a'^2$, and simplifying: $b^2 = 2\,a'^2$. From here we see that $\sqrt{2} =$

b/a'. Hence starting with a fractional representation of $\sqrt{2} = a/b$ we end up with another fractional representation $\sqrt{2} = b/a'$ with a smaller numerator $b < a$. Repeating the same argument with the fraction b/a' we get another fraction with an even smaller numerator, and so on. So the set of possible numerators of a fraction representing $\sqrt{2}$ cannot have a smallest element, contradicting the well-ordering principle. Consequently, our assumption that $\sqrt{2}$ is rational must be false. □