Josephus problem. A group of n people are standing in a circle, numbered consecutively clockwise from 1 to n. Starting with person no. 2, we remove every other person, proceeding clockwise. For example, if $n = 6$, the people are removed in the order 2, 4, 6, 3, 1, and the last person remaining is no. 5. Let $j(n)$ denote the last person remaining. Find some simple way to compute $j(n)$ for any positive integer $n > 1$.

Solution. Note that the problem does not ask for a "simple mathematical formula" for $j(n)$, because the solution is not quite easy to express using only ordinary mathematical symbols, however there is a very simple way to compute $j(n)$ using binary notation:

$$j(n) = \text{ left rotation of the binary digits of } n$$

This means that if $n = x_1x_2x_3 \dots x_n$, where the x_k are the digits of the binary representation of n (with $x_1 \neq 0$) then

$$j(n) = x_2x_3 \dots x_nx_1$$

For instance if $n = 366$ (base 10) = 101101110 (in base 2), then we take the '1' in the left and move it to the right: 011011101, so $j(366) = 011011101$ (base 2) = 111 (base 10).

For instance for $n = 6 = 110$ (base 2) the last person was $j(6) = 5 = 101$ (base 2).

The answer also can be expressed as $j(n) = 2m + 1$, where $m = n - 2^k$, 2^k = maximum power of 2 not exceeding n, i.e., $2^k \leq n < 2^{k+1}$.[1] This can be proved in the following way:

(1) First check that if n = power of 2, say $n = 2^k$, then the last person remaining is always no. 1. This can be proved by induction on k. For $k = 1$ there are only two people, no 2 is removed and no. 1 remains. Then assume that the statement is true for a given k. Assume that $n = 2^{k+1}$. Then people no. $2, 4, 6, \dots$ are removed. After 2^k removals all even numbered people will have been removed, leaving us with exactly the 2^k odd numbered people no. $1, 3, 5, \dots$ By induction hypothesis we know that in this case the first person (i.e. no. 1) remains, so the statement (that no. 1 remains if n = power of 2) is also true for $k + 1$. This completes the induction and proves the statement for every $k \geq 1$.

(2) Finally, if $n = 2^k + m$, where $0 \leq m < 2^k \leq n$, we start by removing the m people numbered $2, 4, 6, \dots, 2m$. Now we have a circle with 2^k people, and the "first one" (which will remain at the end) at that point is no. $2m + 1$.

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[1] Equivalently: $j(n) = 2\,(n - 2^{\lfloor \log_2 n \rfloor}) + 1$, where $\lfloor x \rfloor$ = greatest integer not exceeding x.