Sums and products of tangents squared. Find the values of the following expressions ($n \geq 3$ odd):

$$S(n) = \sum_{k=0}^{\frac{n-3}{2}} \tan^2\left\{\tfrac{(2k+1)\pi}{2n}\right\},$$

$$P(n) = \prod_{k=0}^{\frac{n-3}{2}} \tan^2\left\{\tfrac{(2k+1)\pi}{2n}\right\},$$

$$\frac{S(n)}{P(n)} = \sum_{k=0}^{\frac{n-3}{2}} \prod_{\substack{l=0 \\ l\neq k}}^{\frac{n-3}{2}} \cot^2\left\{\tfrac{(2l+1)\pi}{2n}\right\}.$$

Generalize the result to certain sums of products of tangents or cotangents squared.

Solution. The problem can be solved by looking at the coefficients of the following polynomial (for the second equality use $x^2 - \cot^2\alpha = (x + \cot\alpha)(x - \cot\alpha)$ and recall that $\cot(\pi - \alpha) = -\cot(\alpha)$):

$$p_n(x) = x \prod_{k=0}^{\frac{n-3}{2}} \left(x^2 - \cot^2\left\{\tfrac{(2k+1)\pi}{2n}\right\}\right) = \prod_{k=0}^{n-1} \left(x - \cot\left\{\tfrac{(2k+1)\pi}{2n}\right\}\right).$$

Expanding the product we see that, except for the sign, $1/P(n)$ is the coefficient of x in $p_n(x)$, and $S(n)/P(n)$ is the coefficient of x^3. The other coefficients yield various expressions involving cotangents. So the problem amounts to finding coefficients of $p_n(x)$. To that end we prove that $p_n(x)$ can be rewritten in the following way:

$$p_n(x) = \Re\{(x+i)^n\},$$

where $\Re(z)$ = real part of z (applied to a polynomial we mean the polynomial obtained by replacing the coefficients by their real parts; note that if p is a polynomial with complex coefficients and a is a real number then $\Re\{p\}(a) = \Re\{p(a)\}$). We note that the two polynomials we are comparing have the same degree and same leading coefficient, so we only need to show that they have exactly the same roots, i.e., they are zero for exactly the same values of x. The roots of $p_n(x)$ are $x_k = \cot\left\{\frac{(2k+1)\pi}{2n}\right\}$, so we must prove that $\Re\{(x_k + i)^n\} = 0$, or equivalently $(x_k + i)^n$ is purely imaginary for $k = 0, \dots, n-1$. In fact, putting $\alpha_k = \frac{(2k+1)\pi}{2n}$ we get:

$$\cot\alpha_k + i = \frac{\cos\alpha_k + i\sin\alpha_k}{\sin\alpha_k} = \{\sin\alpha_k\}^{-1} e^{\alpha_k i},$$

so $\arg(\cot\alpha_k + i) = \alpha_k$, and $\arg(\{\cot\alpha_k + i\}^n) = n\alpha_k = (k + 1/2)\pi$, which is the argument of a purely imaginary number.

So now we can expand $(x+i)^n$ using the binomial theorem and find its real part:

$$\Re\{(x_k + i)^n\} = \sum_{l=0}^{\frac{n-1}{2}} (-1)^l \binom{n}{2l} x^{n-2l}.$$

Finally, comparing coefficients we get:

$$\frac{1}{P(n)} = \binom{n}{n-1} = n$$

$$\frac{S(n)}{P(n)} = \binom{n}{n-3} = \frac{n(n-1)(n-2)}{6}$$

hence

$$P(n) = \frac{1}{n}$$

$$S(n) = \frac{(n-1)(n-2)}{6}$$

and much more:

$$\sum_{k=0}^{\frac{n-3}{2}} \cot^2 \left\{ \tfrac{(2k+1)\pi}{2n} \right\} = \binom{n}{2} = \frac{n(n-1)}{2}$$

$$\cot^2\left(\tfrac{\pi}{14}\right)\cot^2\left(\tfrac{3\pi}{14}\right) + \cot^2\left(\tfrac{\pi}{14}\right)\cot^2\left(\tfrac{5\pi}{14}\right) + \cot^2\left(\tfrac{3\pi}{14}\right)\cot^2\left(\tfrac{5\pi}{14}\right) = \binom{7}{4} = 35$$

etc., etc., etc.

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