CHAPTER 1. TOPOLOGY OF ALGEBRAIC VARIETIES, HODGE DECOMPOSITION, AND APPLICATIONS

Contents

In this chapter we will review a number of fundamental facts on the topology of smooth complex projective varieties, and the Hodge decomposition of their singular cohomology with complex coefficients. We will then see them in action by proving the Kodaira Vanishing theorem, the invariance of Hodge numbers under deformations, and the birational invariance of certain Hodge numbers. Some basic references for this material are [GH] Chapter 0 and 1, [La] §3.1 and §4.2, and [Vo].

1. The Lefschetz hyperplane theorem

Theorem 1.1 (**Lefschetz hyperplane theorem).** *Let X be a smooth complex projective variety of dimension n, and let D be an effective ample divisor on X. Then the restriction map*

$$r_i : H^i(X, \mathbf{Z}) \longrightarrow H^i(D, \mathbf{Z})$$

is an isomorphism for $i \le n-2$, and injective for $i = n-1$.

Proof. A conceptual approach is via the following theorem essentially saying that complex affine manifolds have only half as much topology as one might expect:[1]

Theorem 1.2 (**Andreotti-Frankel).** *Let $Y \subset \mathbf{C}^r$ be a closed n-dimensional complex submanifold. Then Y has the homotopy type of a CW complex of real dimension $\le n$. As a consequence*

$$H^i(Y, \mathbf{Z}) = 0 \text{ and } H_i(Y, \mathbf{Z}) = 0 \text{ for } i > n.$$

[1] Note that every $\mathcal{C}^\infty$ manifold of real dimension $2n$ has the homotopy type of a CW complex of real dimension $\le 2n$.

Assuming this for now, let's continue with the proof of the theorem on hyperplane sections. Since D is ample, for some $m \gg 0$ we have that mD is very ample, and therefore there exists an embedding $X \subset \mathbf{P}^N$ and a hyperplane H in $\mathbf{P}^N$ such that $mD = X \cap H$. This implies that $Y = X - D = X - mD$ is a smooth affine complex variety of dimension n. The Andreotti-Frankel theorem implies then that $H_j(Y, \mathbf{Z}) = 0$ for $j > n$. On the other hand, for all j one has by Alexander-Lefschetz duality[2]

$$H_j(Y, \mathbf{Z}) \simeq H^{2n-j}(X, D; \mathbf{Z})$$

and therefore $H^i(X, D; \mathbf{Z}) = 0$ for $i < n$. This is equivalent to the desired conclusion, by the long exact sequence of (relative) cohomology

$$\ldots \longrightarrow H^i(X, D; \mathbf{Z}) \longrightarrow H^i(X, \mathbf{Z}) \longrightarrow H^i(D, \mathbf{Z}) \longrightarrow H^{i+1}(X, D; \mathbf{Z}) \longrightarrow \ldots$$

□

The proof of Theorem 1.2 is a very nice application of basic Morse theory, as in [Mi]. We start by recalling some of its fundamental facts. Let M be a $\mathcal{C}^\infty$ manifold of real dimension n, and let

$$f : M \longrightarrow \mathbf{R}$$

be a $\mathcal{C}^\infty$ function on M. Recall that $p \in M$ is a *critical point* of f if $df_p = 0$, in which case $q = f(p)$ is the corresponding *critical value*. If p is a critical point, then there exists a symmetric bilinear form called the *Hessian* of f at p,

$$\mathrm{Hess}(f)_p = d^2 f_p : T_pX \times T_pX \longrightarrow \mathbf{R},$$

given in local coordinates by $\mathrm{Hess}(f)_p = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$. We say that the p is a *nondegenerate* critical point if $\mathrm{Hess}(f)_p$ is nondegenerate, in which case we define

$$\lambda_p = \mathrm{index}_p(f) = \text{ number of negative eigenvalues of } \mathrm{Hess}(f)_p.$$

The Morse Lemma [Mi] Lemma 2.2 states that in suitable local coordinates around a nondegenerate critical point, f can be written as the quadratic function

$$f(x_1, \ldots, x_n) = -x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_n^2.$$

Theorem 1.3 (**Basic theorem of Morse Theory, [Mi] Theorem 3.5).** *With the notation above, assume that f has the property that $f^{-1}((-\infty, a])$ is compact for every $a \in \mathbf{R}$. Assume in addition that f has only nondegenerate critical points. Then M has the homotopy type of a CW complex with one cell of dimension λ for each critical point of index λ.*

Example 1.4. The height function on the standard two-dimensional torus is a typical example of a function as in Theorem 1.3. The index at each of the critical points can easily be computed to be as in Figure 1. We recover the well known fact that the two-dimensional torus has the homotopy type of a *CW* complex with 1 cell of dimension 0, 2 cells of dimension 1, and 1 cell of dimension 2.

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[2] See e.g. [Hat] §3.3; note that Poincaré duality is the special case $D = 0$.

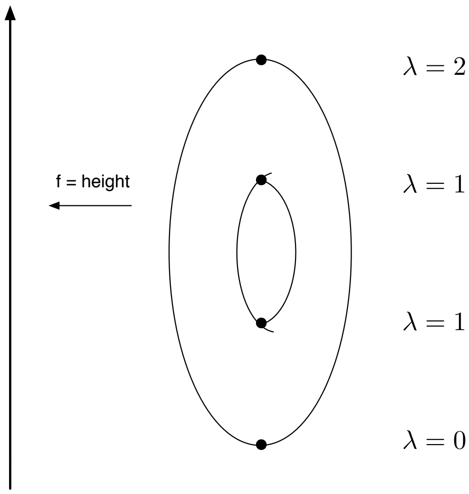

FIGURE 1. Morse theory for the standard torus.

For us, the key example of a function with only nondegenerate critical points (a *Morse function*) is given by the following construction. Assume that $M \subset \mathbf{R}^m$ is a closed submanifold of dimension n. Given $c \in \mathbf{R}^m$, define

$$\varphi_c : M \longrightarrow \mathbf{R}, \; \varphi_c(p) = ||p - c||^2,$$

i.e. the usual Euclidian distance in $\mathbf{R}^m$.

Lemma 1.5 ([Mi] Theorem 6.6)**.** *For almost all $c \in \mathbf{R}^m$, φ_c has only nondegenerate critical points.*

Note that φ_c obviously satisfies the other condition in the statement of Theorem 1.3.

Proof. (of Theorem 1.2.) According to the discussion above, the Theorem reduces to giving an upper bound for the index of a critical point for the distance function on a closed submanifold of $\mathbf{C}^r$. More precisely, it is a consequence of the following result, combined with Theorem 1.3. □

Lemma 1.6. *Let M be an n-dimensional closed complex submanifold of $\mathbf{C}^r = \mathbf{R}^{2r}$, and let $c \in \mathbf{C}^r$. If p is a nondegenerate critical point of φ_c, then $\lambda_p \leq n$.*

Proof. Here is a sketch of an elementary proof, as in the original work of Andreotti-Frankel. (A more conceptual proof can be found for instance in [Mi] §7.) Write $k = r - n$. Choose coordinates on $\mathbf{C}^r$ in such a way that $p = 0$, $c = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the $(n+1)$-st position, and M is the graph of a holomorphic function $f : \mathbf{C}^n \to \mathbf{C}^k$ with $df_0 = 0$. (Exercise: check that you can do this.) In other words, denoting the coordinates by $z_1, \ldots, z_n$, we have

$$M = \{\big(z_1, \ldots, z_n, f_1(z), \ldots, f_k(z)\big) \mid f_1, \ldots, f_k \text{ holomorphic with } \mathrm{ord}_0(f_i) \geq 2\}.$$

The distance function φ_c is then given by the formula

$$\varphi_c(z) = (1 - 2 \cdot \text{Re } f_1(z)) + \sum_{i=1}^{n} |z_i|^2 + \sum_{i=2}^{k} |f_i(z)|^2.$$

Since $\text{ord}_0(f_i) \geq 2$ for all i, the last sum in the formula does not contribute to $\text{Hess}(\varphi_c)_0$. Now write

$$f_1(z) = Q(z) + \text{ terms of order } \geq 3,$$

where $Q(z)$ is a homogeneous quadratic polynomial in $z_1, \ldots, z_n$. Putting all of this together we get

$$\text{Hess}(\varphi_c)_0 = -2 \cdot \text{Hess}(\text{Re } Q(z))_0 + 2 \cdot \text{Id}.$$

As the second term is positive definite, the result follows from the following standard

Lemma 1.7. *If Q is a complex homogeneous quadratic polynomial in $z_1, \ldots, z_n$, then* $\text{Hess}(\text{Re } Q(z))_0$ *has at most n positive and at most n negative eigenvalues.*

Proof. After a complex change of coordinates $z \to w$ one can write

$$Q(w) = w_1^2 + \ldots + w_s^2$$

with $s \leq n$. Writing $w_j = x_j + i \cdot y_j$, we have

$$\text{Re } Q(w) = (x_1^2 - y_1^2) + \ldots + (x_s^2 - y_s^2),$$

and for this the statement is clear. □

□

2. The Hodge decomposition

For a smooth complex projective variety, or more generally a compact Kähler manifold, a fundamental result is the Hodge decomposition of its singular cohomology with complex coefficients.

Theorem 2.1 (Hodge decomposition). *If X is a compact Kähler manifold, then there is a decomposition*

$$H^i(X, \mathbf{C}) \simeq \bigoplus_{p+q=i} H^{p,q}(X)$$

with:

- $H^{p,q}(X) = \overline{H^{q,p}(X)}$
- $H^{p,q}(X) \simeq H^q(X, \Omega_X^p)$.

As $H^{p,q}$ usually denotes the cohomology group obtained from (p,q)-forms via the $\overline{\partial}$ operator, the second bullet is in fact the independent *Dolbeaut Theorem*. In particular, the $H^{q,0}(X)$ spaces can be seen as the global sections of the various bundles of holomorphic forms on X, or after conjugation as the cohomology groups of the structure sheaf. As such, we will see for instance that they are birational invariants.

We use the standard notation

$$b_i(X) = \dim_{\mathbf{C}} H^i(X, \mathbf{C}),$$

the i-th *Betti number* of X, and

$$h^{p,q}(X) = \dim_{\mathbf{C}} H^{p,q}(X) = \dim_{\mathbf{C}} H^q(X, \Omega_X^p) = \dim_{\mathbf{C}} H^p(X, \Omega_X^q),$$

the (p,q)-*Hodge number* of X. By the Hodge decomposition theorem we have

$$b_i(X) = \sum_{p+q=i} h^{p,q}(X) \text{ and } h^{p,q}(X) = h^{q,p}(X).$$

As a first example of application, this gives an immediate obstruction to a complex manifold being Kähler:

Corollary 2.2. *If X is compact Kähler and k is an odd integer, then $b_k(X)$ is even.*

Given the Hodge decomposition, there is a holomorphic version of the Lefschetz Hyperplane Theorem 1.1:

Corollary 2.3. *Let X be a smooth complex projective variety of dimension n, and let D be a smooth effective ample divisor on X. Then the restriction maps*

$$r_{p,q} : H^q(X, \Omega_X^p) \longrightarrow H^q(D, \Omega_D^p)$$

are isomorphisms for $p + q \leq n - 2$, and injective for $p + q = n - 1$.

Proof. The restriction maps naturally commute with the functorial decomposition given by the Hodge theorem, so that $r_i = \oplus_{p+q=i} r_{p,q}$. The statement follows then immediately from Theorem 1.1. □

Definition 2.4 (Hodge diamond)**.** We usually collect the Hodge numbers of a compact Kähler manifold X in a *Hodge diamond*, as represented below.

$$\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & \vdots & & \ddots & \\
h^{n,0} & h^{n-1,1} & & \cdots & & h^{1,n-1} & h^{0,n} \\
 & \ddots & & \vdots & & & \\
 & & h^{n,n-1} & & h^{n-1,n} & & \\
 & & & h^{n,n} & & &
\end{array}$$

This diamond has a few symmetries. Conjugation $H^{p,q} = \overline{H^{q,p}}$ implies that it is invariant under reflection across the middle column. Serre duality gives the isomorphism $H^{p,q} \simeq H^{n-p,n-q*}$, which gives diagonal symmetry (in other words the Hodge diamond is left invariant under rotation by 180°). This properties imply invariance under reflection across the middle row as well; this isomorphism can also be seen directly via the Hodge $*$-operator.

Exercise 2.5. Find the Hodge diamonds of the following varieties:

(i) $\mathbf{P}^n$.

(ii) a smooth degree 3 hypersurface in $\mathbf{P}^3$ (cubic surface).

(iii) a smooth degree 3 hypersurface in $\mathbf{P}^4$ (cubic threefold).

One Hodge space that has a special interpretation is $H^{1,1}(X)$. Every class here is known to be analytic, in other words it comes from a divisor on X. For a proof of the theorem below see e.g. [GH] p.163.

Theorem 2.6 (**Lefschetz theorem on** $(1,1)$**-classes**)**.** *Let X be a smooth projective complex variety of dimension n. Then every class in $H^{1,1}(X) \cap H^2(X, \mathbf{Z})$ (i.e. integral $(1,1)$-class) is the first Chern class of a line bundle on X, or equivalently the Poincaré dual of $[D] \in H_{2n-2}(X, \mathbf{Z})$ for some divisor D on X.*

3. Hodge numbers in smooth families

Definition 3.1 (**Families**)**.** A holomorphic map $\pi : \mathcal{X} \to B$ between complex manifolds is called a *family of complex manifolds* if π is a proper submersion. If $\mathcal{X}$ and B are smooth varieties, we can express this by saying that π is a smooth proper morphism (in the algebraic sense). For $b \in B$, we denote by $X_b = \pi^{-1}(b)$, seen as a complex manifold (or smooth algebraic variety).

From the topological, or even differential, point of view, the fibers of a family as above over a contractible base (for instance a disk) are all the same. For a proof of the following theorem, and for further extensions, see e.g. [Vo] §9.1.

Theorem 3.2 (**Ehresmann**)**.** *Let $\pi : \mathcal{X} \to B$ be a proper submersion between smooth manifolds, with B contractible. Consider a base point $b_0 \in B$. Then there exists a diffeomorphism*

$$\Phi : \mathcal{X} \longrightarrow X_{b_0} \times B$$

relative to B, i.e. such that $\pi = p_2 \circ \Phi$.

Remark 3.3. The statement of the theorem is far from being true if one requires the trivialization Φ to be holomorphic (or algebraic) rather than just $\mathcal{C}^\infty$, hence the theory of moduli in these more restrictive categories. On the other hand, the diffeomorphisms $X_b \to X_{b_0}$ induced by Φ for any $b \in B$ enable us to see the family as given by different complex structures varying with b, on the fixed differentiable manifold X_{b_0}.

As any manifold has by definition an open cover with contractible submanifolds, for which the previous theorem applies, we obtain that the topology of the fibers of a family is the same. In particular:

Corollary 3.4. *If $\pi : \mathcal{X} \to B$ is a proper submersion of smooth manifolds, then*

$$H^i(X_{b_1}, \mathbf{Z}) \simeq H^i(X_{b_2}, \mathbf{Z})$$

for any integer i and any $b_1, b_2 \in B$.

Theorem 3.5. *Let $\pi : \mathcal{X} \to B$ be a smooth family of complex projective varieties (or compact Kähler manifolds). Then the Hodge numbers $h^{p,q}(X_b)$ are constant for $b \in B$.*

Proof. Note that by the Dolbeaut isomorphism, we can see the Hodge numbers as representing the dimension of various cohomology groups of holomorphic vector bundles:

$$h^{p,q}(X) = \dim_{\mathbf{C}} H^q(X, \Omega_X^p).$$

But by the Semicontinuity theorem, these vary upper-semicontinuously when we move b. (Indeed, one can apply the Semicontinuity theorem to the bundle of relative differentials $\Omega^p_{\mathcal{X}/B}$ on $\mathcal{X}$, whose restriction to each X_b is $\Omega^p_{X_b}$.) In other words, if we fix any point $b_0 \in B$, we have for all p and q that

$$h^{p,q}(X_b) \le h^{p,q}(X_{b_0})$$

for b in a neighborhood of b_0. This implies for each i, by the Hodge decomposition, that

$$b_i(X_b) = \sum_{p+q=i} h^{p,q}(X_b) \le \sum_{p+q=i} h^{p,q}(X_{b_0}) = b_i(X_{b_0})$$

and since the Betti numbers on the two extremes are equal according to Corollary 3.4, we obtain that all Hodge numbers are constant in a neighborhood of b_0. Now cover the base B with sufficiently small open subsets on which the above applies. □

Remark 3.6. A more general statement holds in the setting of compact complex manifolds (see e.g. [Vo] Proposition 9.20): let $\pi : \mathcal{X} \to B$ be a family (i.e. a proper submersion) of compact complex manifolds such that X_{b_0} is Kähler for some $b_0 \in B$. Then for b in a neighborhood of b_0 we have

$$h^{p,q}(X_b) = h^{p,q}(X_{b_0}) \text{ for all } p \text{ and } q,$$

and moreover the Hodge to de Rham spectral sequence degenerates at E_1. In fact one can also show that X_b must be Kähler for b sufficiently close to b_0 (see [Vo] Theorem 9.23).

4. Birationally invariant Hodge numbers

Let X be a smooth projective variety of dimension n over an algebraically closed field. It is an elementary result that the Hodge numbers

$$h^{p,0}(X) = h^0(X, \Omega_X^p)$$

are birational invariants for all q.

Proposition 4.1. *If X and Y are birational smooth projective varieties over an algebraically closed field, then*

$$h^{p,0}(X) = h^{p,0}(Y) \text{ for all } p.$$

Proof. By symmetry, it is enough to show that $h^{p,0}(Y) \le h^{p,0}(X)$. Let f be a (bi)rational map from X to Y, $V \subset X$ the maximal open set on which f is defined, and $U \subset V$ an open subset on which the induced $f : U \to f(U)$ is an isomorphism.

By pulling back p-forms via the morphism $f : V \to Y$, we get an induced map

$$f^* : H^0(Y, \Omega_Y^p) \longrightarrow H^0(V, \Omega_V^p).$$

The first claim is that this map is injective; indeed via f we get an isomorphism $\Omega^p_{V|U} \simeq \Omega^p_{Y|f(U)}$. If f^* were not injective, it would mean that a nonzero section in $H^0(Y, \Omega^p_Y)$ would vanish on a nonempty open set, which is a contradiction.

The second claim is that the restriction map

$$H^0(X, \Omega^p_X) \longrightarrow H^0(V, \Omega^p_V)$$

is an isomorphism, which combined with the above finishes the proof. But it is well-known that any rational map on a smooth (or more generally normal) variety is defined in codimension 1, i.e. $\mathrm{codim}_X(X - V) \geq 2$. Now by Hartogs' theorem (or its algebraic analogue) any regular function, hence any regular q-form, extends over a codimension 2 subset. This implies the surjectivity of the restriction map, while injectivity follows as in the paragraph above. □

Exercise 4.2. Check the assertion above, namely that $\mathrm{codim}_X(X - V) \geq 2$, by using the valuative criterion of properness. (Cf. also [Har] II Ex. 3.20 and III Ex. 3.5 for the case of rational functions, referred to as *Hartogs' theorem* in the proof of the Proposition.)

Remark 4.3. If X is defined over (a subfield of) the complex numbers, then by Hodge duality we have $h^{p,0} = h^{0,p}$, hence Proposition 4.1 automatically implies that if X and Y are birational then

$$h^p(X, \mathcal{O}_X) = h^p(Y, \mathcal{O}_Y).$$

This statement is still true in arbitrary characteristic, but the argument above does not work any more. Instead, one can use the following fundamental result on birational morphisms:[3]

Theorem 4.4. *Let $f : X \to Y$ be a birational morphism between smooth varieties. Then*

$$f_*\mathcal{O}_X \simeq \mathcal{O}_Y \text{ and } R^i f_*\mathcal{O}_X = 0 \text{ for } i > 0.$$

This is well-known (but nontrivial) in characteristic 0, using fundamental facts on resolution of singularities. If resolution were known in characteristic $p > 0$, the argument would go through; at the moment this is not the case. However, the statement above was recently proved, with different methods, by Chatzistamatiou-Rülling [CR].

Going back to the proof of our invariance, since X and Y are birational, by the general resolution of singularities machinery they are dominated by a common model, i.e. there exists a smooth projective Z and surjective birational morphisms

$$f : Z \longrightarrow X \ \text{ and } \ g : Z \longrightarrow Y.$$

We can therefore assume that there is a birational *morphism* $f : X \to Y$. But by Theorem 4.4, the Leray spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_*\mathcal{O}_X) \Rightarrow H^{p+q}(X, \mathcal{O}_X)$$

degenerates at E_2, giving isomorphisms

$$H^p(Y, \mathcal{O}_Y) \simeq H^p(X, \mathcal{O}_X).$$

[3] This result could be phrased as saying that smooth varieties have rational singularities.

Example 4.5. It is not true in general that all Hodge numbers are birational invariants. The simplest example is that of blow-ups. Let X be a smooth projective variety (or more generally complex manifold) of dimension n, and let $\tilde{X} = \mathrm{Bl}_x(X)$ be the blow-up of X at an arbitrary point x. Denote by $E \simeq \mathbf{P}^{n-1}$ the exceptional divisor. This divisor introduces its own new cycles (including E itself) to $H_i(\tilde{X}, \mathbf{Z})$. More precisely, for each $i > 0$ one has

$$H_i(\tilde{X}, \mathbf{Z}) \simeq H_i(X, \mathbf{Z}) \oplus H_i(E, \mathbf{Z}).$$

This reflects in the Hodge decomposition. Since all analytic cycles are in $H^{p,p}$ for various p, we obtain

$$H^{p,p}(\tilde{X}) \simeq H^{p,p}(X) \oplus \mathbf{C}$$

for all $0 < p < n$, so the non-extremal Hodge numbers on the middle column go up by 1.

Exercise 4.6. Verify carefully all the statements made in the Example above.

5. The topological approach to the Kodaira vanishing theorem

My emphasis in these notes is on results directly focused on the singular cohomology and Hodge numbers of smooth projective varieties. However, here I take a moment to exemplify how such results can be applied in order to derive statements of fundamental importance in birational geometry.

Concretely, we will use the previous topological and Hodge theoretic results in this chapter, together with a covering construction, in order to derive the celebrated Kodaira Vanishing theorem. That this is possible was first observed by Ramanujam; Kodaira's original proof was of a more differential geometric nature (cf. [GH] Ch.I §2). We follow the treatment in [La] §4.2. This approach has led to numerous important generalizations of the Kodaira Vanishing theorem (see for instance [EV]).

Theorem 5.1 (**Kodaira Vanishing Theorem).** *Let X be a smooth complex projective variety of dimension n, and let D be an ample divisor on X. Then*

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0 \text{ for all } i > 0.$$

Equivalently,

$$H^i(X, \mathcal{O}_X(-D)) = 0 \text{ for all } i < n.$$

Proof. **Step 1.** We first prove the theorem in the case when D is a *smooth effective* divisor. In this case we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$$

which induces the long exact sequence on cohomology

$$\cdots \longrightarrow H^{i-1}(X, \mathcal{O}_X) \longrightarrow H^{i-1}(D, \mathcal{O}_D) \longrightarrow H^i(X, \mathcal{O}_X(-D)) \longrightarrow$$
$$\longrightarrow H^i(X, \mathcal{O}_X) \longrightarrow H^i(D, \mathcal{O}_D) \longrightarrow \cdots$$

But the case $p = 0$ and $q = i$ in Corollary 2.3 says that the restriction maps

$$H^j(X, \mathcal{O}_X) \longrightarrow H^j(D, \mathcal{O}_D)$$

are isomorphisms for $j \leq n - 2$ and injective for $j = n - 1$, which using the sequence above implies $H^i(X, \mathcal{O}_X(-D)) = 0$ for $i < n$.

Step 2. We now reduce to the case treated in Step 1 by means of a standard *cyclic covering* construction. Since D is ample, for $m \gg 0$ there is a smooth irreducible divisor $B \in |mD|$. Proposition 5.7 below implies that we can take an m-th root of this divisor, at the expense of working on a finite cover of X. Concretely, consider $f : Y \to X$ to be the m-fold cyclic cover branched along B. Denote $D' = f^*D$. Proposition 5.7 says that Y is smooth and that there exists a smooth effective divisor $B' \in |D'|$, which is obviously ample. We now claim that in order to conclude it suffices to have

$$H^i(Y, \mathcal{O}_Y(-B')) = 0 \text{ for all } i < n,$$

which holds by the previous step. Indeed, we have by definition

$$\mathcal{O}_Y(-B') = \mathcal{O}_Y(-D') \simeq f^*\mathcal{O}(-D)$$

and so the claim follows from Lemma 5.8 below. □

One can in fact use a similar argument to prove the more general Nakano Vanishing, a statement about arbitrary bundles of holomorphic forms (Kodaira Vanishing is the special case $p = n$).

Theorem 5.2 (Nakano Vanishing Theorem). *Let X be a smooth complex projective variety, and L an ample line bundle on X. Then*

$$H^q(X, \Omega_X^p \otimes L) = 0 \;\text{ for } p + q > n,$$

or equivalently

$$H^q(X, \Omega_X^p \otimes L^{-1}) = 0 \;\text{ for } p + q < n.$$

Before proving the Theorem, we need to introduce one more construction:

Definition 5.3 (Forms with log-poles)**.** Let X be a smooth variety, and D a smooth effective divisor on X. The sheaf of 1-forms on X with log-poles along D is

$$\Omega_X^1(\log\, D) = \Omega_X^1 < \frac{df}{f} >, \;f \text{ local equation for } D.$$

Concretely, if $z_1, \ldots, z_n$ are local coordinates on X, chosen such that $D = (z_n = 0)$, then $\Omega_X^1(\log\, D)$ is locally generated by $dz_1, \ldots, dz_{n-1}, \frac{dz_n}{z_n}$. This is a free system of generators, so $\Omega_X^1(\log\, D)$ is locally free of rank n. For any integer p, we define

$$\Omega_X^p(\log\, D) := \bigwedge^p \big(\Omega_X^1(\log\, D)\big).$$

Lemma 5.4. *There are short exact sequences:*

(i) $0 \longrightarrow \Omega_X^p \longrightarrow \Omega_X^p(\log\, D) \longrightarrow \Omega_D^{p-1} \longrightarrow 0.$

(ii) $0 \longrightarrow \Omega_X^p(\log\, D)(-D) \longrightarrow \Omega_X^p \longrightarrow \Omega_D^p \longrightarrow 0.$

Proof. I will sketch the proof for $p = 1$; in general it is only notationally more complicated. The comprehensive source for this is [EV] §2.

Choose local analytic coordinates $z_1, \ldots, z_n$ so that $D = (z_n = 0)$. For (i), the map on the right is the *residue map* along D

$$\mathrm{res_D} : \Omega^1_X(\log\, D) \longrightarrow \mathcal{O}_D$$

given by

$$f_1 dz_1 + \ldots + f_{n-1} dz_{n-1} + f_n \frac{dz_n}{z_n} \mapsto f_{n|D},$$

where $f_1, \ldots, f_n$ are local functions on X. The right hand side is 0 if one can write $f = z_n \cdot g$ for an arbitrary regular function g. Therefore we can see the kernel as being locally generated by $dz_1, \ldots, dz_n$, hence isomorphic to Ω^1_X.

For (ii), the map on the right is given by restriction of forms. Since locally $D = (z_n = 0)$, the kernel of the restriction map $\Omega^1_X \to \Omega^1_D$ is locally generated by $z_n dz_1, \ldots z_n dz_{n-1}, dz_n$. But these obviously generate the subsheaf $\Omega^p_X(\log\, D)(-D) \subset \Omega^p_X(\log\, D)$. □

Lemma 5.5. *Let $f : Y \to X$ be the m-fold cyclic cover branched along D, as in Proposition 5.7. Let D' be the divisor in Y such that $f^*D = mD'$, mapping isomorphically onto D. Then*

$$f^*\Omega^p_X(\log\, D) \simeq \Omega^p_Y(\log\, D').$$

Proof. It is enough to prove this for $p = 1$. As usual, choose local coordinates $z_1, \ldots, z_n$ so that $D = (z_n = 0)$. We then have local coordinates $z_1, \ldots, z_{n-1}, w$ on Y such that $D' = (w = 0)$ and f is given by $z_n = w^m$. Then we see that

$$f^*(\frac{dz_n}{z_n}) = m \cdot \frac{dw}{w},$$

which implies what we want. □

Proof of Theorem 5.2. We will show the second version of the statement of the Theorem. For $m \gg 0$, let $D \in |mL|$ be a smooth divisor. Let $f : Y \to X$ be the m-fold cyclic cover branched along D as in Proposition 5.7, with $f^*D = mD'$ and $L' = \mathcal{O}_Y(D')$.

We can assume by induction on $n = \dim X$ that we already know Nakano vanishing on D, so that

$$H^q(D, \Omega^{p-1}_D \otimes L^{-1}_{|D}) = 0 \text{ for } p + q < n.$$

Using this and passing to cohomology in the sequence in Lemma 5.4(i), it suffices then to prove that

$$H^q(X, \Omega^p_X(\log\, D) \otimes L^{-1}) = 0 \text{ for } p + q < n.$$

Now using Lemma 5.5 together with Lemma 5.8 below, this is equivalent with proving

$$H^q(Y, \Omega^p_Y(\log\, D') \otimes \mathcal{O}_Y(-D')) = 0 \text{ for } p + q < n.$$

Finally, we appeal to the exact sequence in Lemma 5.4(ii). Using this, our desired statement is equivalent to the fact that the restriction maps

$$r_{p,q} : H^q(Y, \Omega^p_Y) \longrightarrow H^q(D', \Omega^p_{D'})$$

are isomorphisms for $p + q \leq n - 2$, and injective for $p + q = n - 1$. But this is precisely the statement of the holomorphic Lefschetz hyperplane theorem, Corollary 2.3. □

Remark 5.6. Note that one can completely reverse the argument above, and deduce the Lefschetz hyperplane theorem from Nakano vanishing (which in turn can be proved by various other methods); many references, especially of a more differential geometric nature, follow that approach.

Appendix: Covering Lemmas. I will state here without proof a useful technical result needed in order to "take m-th roots" of divisors $B \in |mD|$ as in the previous section. This is only the tip of the iceberg, as more complicated constructions are needed for deeper applications. For a thorough survey and clean proofs see [La] §4.1.B.

Proposition 5.7. *Let X be a variety over an algebraically closed field k, and let L be a line bundle on X. Let $0 \neq s \in H^0(X, L^{\otimes m})$ for some $m \geq 1$, with $D = Z(s) \in |mL|$. Then there exists a finite flat morphism $f : Y \to X$, where Y is a scheme over k such that if $L' = f^*L$, there is a section*

$$s' \in H^0(Y, L') \text{ satisfying } (s')^m = f^*s.$$

Moreover:

- *if X and D are smooth, then so are Y and $D' = Z(s')$.*
- *the divisor D' maps isomorphically onto D.*
- *there is a canonical isomorphism $f_*\mathcal{O}_Y \simeq \mathcal{O}_X \oplus L^{-1} \oplus \cdots \oplus L^{-(m-1)}$.*

The scheme Y in Proposition 5.7 is called the *m-fold cyclic cover of X branched along D*. We also need to compare cohomology via finite covers.

Lemma 5.8. *Let $f : Y \to X$ be a finite surjective morphism of normal complex varieties, and let $\mathcal{E}$ be a locally free sheaf on X. If for some $i \geq 0$ one has $H^i(Y, f^*\mathcal{E}) = 0$, then $H^i(X, \mathcal{E}) = 0$.*

Exercise 5.9. Prove Lemma 5.8.

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