VANISHING THEOREMS FOR TOPOLOGICALLY TRIVIAL LINE BUNDLES AND APPLICATIONS

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I will survey a very influential part of Rob’s work, namely **Generic Vanishing theory**. It was introduced by Rob together with Mark Green in the late 80’s and early 90’s, and has since become indispensable in studying the geometry of varieties with non-zero 1st Betti number (and beyond). Will describe:

- The initial package.
- Some of the most striking applications over the years.
- Recent developments and extensions of the theory, and new applications they facilitated.
Will always denote $X = \text{smooth projective variety over } \mathbb{C}$ (although much of the general theory works for compact Kähler manifolds). Say $\dim X = n$.

As motivation, recall \textit{Kodaira Vanishing}: if $L$ ample on $X$, then

$$H^i(X, \omega_X \otimes L) = 0, \text{ for all } i > 0.$$ 

This is well known to have myriads of applications, and lots of useful generalizations.

\textbf{Question.} \textit{What can one say under positivity hypotheses that are milder than the ampleness of $L$? For instance, how about $\omega_X$ itself? (Think $\omega_X = \omega_X \otimes \mathcal{O}_X$...)}
Examples. (1) Let $X = C$ = smooth projective curve of genus $g \geq 1$. Then

$$H^1(C, \omega_C) \simeq \mathbb{C} \neq 0$$

but

$$H^1(C, \omega_C \otimes L) = 0, \quad \text{for all } L \in \text{Pic}^0(C) - \{0\}.$$

(Keep in mind $C \hookrightarrow J(C)$.)

(2) Let $X = S \xrightarrow{f} C$ be an elliptic surface over a curve of genus $g \geq 2$. For $L \in \text{Pic}^0(S)$, as above

$$H^2(S, \omega_S \otimes L) \neq 0 \iff L = \mathcal{O}_S.$$

However, the picture for $H^1$ is interesting; for $L \in \text{Pic}^0(C)$, one can use:

$$H^1(S, \omega_S \otimes f^*L) \simeq H^1(C, f_*\omega_S \otimes L) \oplus H^0(C, \omega_C \otimes L) \neq 0.$$

• $f$ non-isotrivial: $\text{Pic}^0(S) = f^*\text{Pic}^0(C) = \{M \mid H^1(S, \omega_S \otimes M) \neq 0\}$ (So Albanese map contracts fibers of $f$, and no vanishing whatsoever.)

• $f$ isotrivial: this time

$$f^*\text{Pic}^0(C) \subseteq \{M \mid H^1(S, \omega_S \otimes M) \neq 0\} \not\subseteq \text{Pic}^0(S).$$

(No Albanese map generically finite, and $H^1$ vanishes generically.)
For any $i$ define the algebraic subset
\[ V^i(\omega_X) := \{ L \in \text{Pic}^0(X) \mid H^i(X, \omega_X \otimes L) \neq 0 \} \subset \text{Pic}^0(X), \]
called the \textit{i-th cohomological support locus} of $\omega_X$. Other examples of this sort, together with some conjectures of Beauville and Catanese led Green and Lazarsfeld to study, dually, the deformation problem for cohomology groups of topologically trivial line bundles:
\[ H^i(X, L) \text{ with } L \in \text{Pic}^0(X). \]
This in turn led to
Generic Vanishing Package:

**Dimension Theorem.** (Green-Lazarsfeld, ’87)
Consider the Albanese map $a : X \to \text{Alb}(X)$. Then
\[ \text{codim } V^i(\omega_X) \geq i - \dim X + \dim a(X) \text{ for all } i. \]

**Linearity Theorem.** (Green-Lazarsfeld, ’91; Arapura, Simpson ’93)
The irreducible components of each $V^i(\omega_X)$ are torsion translates of abelian sub-varieties of $\text{Pic}^0(X)$.

**Strong Linearity Theorem.** (Green-Lazarsfeld, ’91)
I have to sacrifice something...

**Fibrations.** (Green-Lazarsfeld, ’91)
Let $W$ be a positive dimensional irreducible component of $V^i(\omega_X)$ for some $i$. Then there exists a morphism $f : X \to Y$, with $Y$ normal, of maximal Albanese dimension, and $\dim Y \leq n - i$, such that
\[ W \subseteq \tau + f^*\text{Pic}^0(Y), \quad \tau \in \text{Pic}^0(X). \]
**Basic example.** A variety is called of *maximal Albanese dimension* if its Albanese map is generically finite onto its image, i.e. \( \dim X = \dim a(X) \). In this case the Dimension Theorem says that \( V^i(\omega_X) \) are proper subsets of \( \text{Pic}^0(X) \) as soon as \( i > 0 \), so as a weaker statement we see the first instance of a “generic” analogue of Kodaira vanishing:

\[
H^i(X, \omega_X \otimes L) = 0, \quad \text{for all } i > 0 \text{ and } L \in \text{Pic}^0(X) \text{ general.}
\]

This means that sometimes Generic Vanishing can be applied similarly to standard vanishing theorems; for instance, if \( X \) is of maximal Albanese dimension, then

\[
\chi(\omega_X) = \chi(\omega_X \otimes L) = h^0(X, \omega_X \otimes L) \geq 0
\]

\[
\uparrow \quad \text{any } L \quad \text{general } L
\]
Some of the first important applications:

- paracanonical systems
- singularities of Theta divisors
- a characterization of abelian varieties
- pluricanonical systems on varieties of maximal Albanese dimension
- the list is by no means exhaustive...
Singularities of theta divisors. The initial result that really showed the type of possible birational geometry applications:

**Theorem.** (Ein-Lazarsfeld, ’97) If \((A, \Theta)\) is an irreducible principally polarized abelian variety, then \(\Theta\) is normal and has rational singularities.

**Idea:** Use the adjoint ideal \(\text{Adj}(\Theta)\), i.e. an ideal sheaf on \(A\) sitting in an exact sequence

\[
0 \longrightarrow \mathcal{O}_A \longrightarrow \mathcal{O}_A(\Theta) \otimes \text{Adj}(\Theta) \longrightarrow f_*\omega_X \longrightarrow 0,
\]

where \(f : X \to \Theta\) is a resolution of singularities. It is a fact that \(\Theta\) satisfies the required conclusion if and only if \(\text{Adj}(\Theta) \cong \mathcal{O}_A\). Method:

- Twist the sequence by general \(L \in \text{Pic}^0(A)\).
- Apply the Dimension Theorem to \(\omega_X\) (\(X\) is of maximal Albanese dimension).
- Conclude \(\text{Adj}(\Theta) \cong \mathcal{O}_A\) since its zero locus has to satisfy \(Z \subset \bigcap_{a \in A} (\Theta + a) = \emptyset\).

Characterization of abelian varieties. A conjecture of Kollár improving previous numerical characterizations of abelian varieties:

**Theorem.** (Chen-Hacon, 2001) A smooth projective variety is birational to an abelian variety if and only if \(q(X) = \dim X\) and \(P^1(X) = P^2(X) = 1\).

**Notation:** \(q(X) = h^0(X \to \mathcal{O}_X); P^m(X) = h^0(X \to \mathcal{O}_X^m)\).

**Idea:**

- Show that 0 is an isolated point in \(V_0(\omega_X)\) by using \(GV\) methods to see that otherwise there is a positive dimensional component 0 such that \(Z \subset V_0(\omega_X)\) as well, contradiction with \(P^2(X) = 1\).
- Use result of Ein-Lazarsfeld proved in a different context, saying that if 0 is isolated, then \(X\) is birational to an abelian variety.

Effective results on pluricanonical maps. By a result of Hacon-McKernan, Takayama, and Tsuji, for every dimension \(n\) there exists \(f(n)\) such that the linear system \(|mK_X|\) gives birational map for \(m\geq f(n)\). However, for varieties of maximal Albanese dimension the situation is much better (for an analogy, think of the Lefschetz theorem on abelian varieties):

**Theorem.** (Chen-Hacon, ’01; Jiang-Lahoz-Tirabassi, ’11) Let \(X\) be of maximal Albanese dimension. If \(X\) is of general type, then \(|3K_X|\) is birational. If not, \(|4K_X|\) gives the Iitaka fibration.

Barja-Lahoz-Naranjo-Pareschi, ’10: strong results towards the classification of those varieties of general type and maximal Albanese dimension for which \(|2K_X|\) is not birational.
Characterization of abelian varieties. A conjecture of Kollár improving previous numerical characterizations of abelian varieties:

**Theorem.** (Chen-Hacon, ’01) A smooth projective variety is birational to an abelian variety if and only if $q(X) = \dim X$ and $P_1(X) = P_2(X) = 1$.

Notation: $q(X) = h^0(X, \Omega^1_X); P_m(X) = h^0(X, \omega_X^\otimes m)$.

**Idea:** • Show that 0 is an isolated point in $V^0(\omega_X)$: use GV methods to see that otherwise there is a positive dimensional component $0 \in Z$ such that $-Z \subset V^0(\omega_X)$ as well, contradiction with $P_2(X) = 1$.

• Use result of Ein-Lazarsfeld proved in a different context, saying that if $0 \in V^0(\omega_X)$ is isolated, then $X$ is birational to an abelian variety.
Effective results on pluricanonical maps. By a result of Hacon-McKernan, Takayama, and Tsuji, for every dimension $n$ there exists $f(n)$ such that the linear system $|mK_X|$ gives birational map for $m \geq f(n)$. However, for varieties of maximal Albanese dimension the situation is much better:

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Barja-Lahoz-Naranjo-Pareschi, ’10: results towards classifying varieties of general type and maximal Albanese dimension for which $|2K_X|$ is not birational.

Idea: Use generic vanishing to deduce the continuous global generation of sheaves of the form

$$O_X(mK_X) \otimes \mathcal{I}(X, \| (m - 1)K_X \|)$$

and of related gadgets, where the ideal above is an asymptotic multiplier ideal. Then generically apply a global generation criterion for varieties that are finite over abelian varieties coming from the theory of $M$-regularity (Pareschi – P.).
Recent developments:

• Generic Vanishing theory has been strengthened by homological and Hodge theoretic methods; new classes of applications.

• (1) Derived category approach (Hacon, 2003): allows for substantial extension of the theory. Hacon used the Fourier-Mukai transform, duality and commutative algebra to reduce the Dimension Theorem to Kawamata-Viehweg vanishing.
Consider $P$ a Poincaré line bundle on $X \times \hat{A}$. Define integral functor

$$ \Phi_P : D(X) \to D(\hat{A}), \quad \mathcal{F} \mapsto R^{p_2*}(p_1^* \mathcal{F} \otimes P). $$

Hacon showed using vanishing that

\[(1) \quad \Phi_P(\mathcal{O}_X) \in D^{[\dim a(X), n]}(\hat{A}). \]

In particular, if $X$ is of maximal Albanese dimension, it is a sheaf in degree $n$, i.e.

$$ R^i p_{2*} P = 0, \quad \text{for all } i \neq n, $$

a conjecture of Green-Lazarsfeld, and this implies the Dimension Theorem.
Opened the door for putting Generic Vanishing in a larger picture, formalizing it as a property satisfied by many objects (GV-sheaves) with respect to integral functors (Pareschi – P., ’08–’10).

- relationship with perverse coherent sheaves + other moduli spaces...
- Dimension Theorem really does complement Kodaira vanishing: it is satisfied for instance by $\omega_X \otimes L$ with arbitrary $L$ nef.
- Local study of the GV condition; e.g. Green-Lazarsfeld conjecture in the Kähler case.
Numerical inequalities for irregular varieties. (1) Higher dimensional Castelnuovo-de Franchis theorem.

**Theorem.** (Pareschi – P., ’09) If $X$ is a smooth projective irregular variety without irregular fibrations, i.e. with no morphisms $f : X \to Y$ with $Y$ normal and of maximal Albanese dimension, $0 < \dim Y < \dim X$, then

$$\chi(\omega_X) \geq q(X) - \dim X.$$ 

This is an extension to arbitrary dimension of the classical Castelnuovo-de Franchis inequality $p_g \geq 2q - 3$ (which can be rewritten as $\chi(\omega_X) \geq q(X) - 2$) for surfaces without irrational pencils.
Idea: Hypothesis implies that $X$ is of maximal Albanese dimension, so we have seen that in any case $\chi(\omega_X) \geq 0$. Point of the Theorem is that we can be more precise. Define the Generic Vanishing index of $X$ as

$$\text{gv}(X) := \min_{i \geq 0} \{\text{codim } V^i(\omega_X) - i\} \geq 0.$$ 

The Theorem really is

- $X$ of maximal Albanese dimension $\implies \chi(\omega_X) \geq \text{gv}(X)$.

To show this, GV theory implies that the Fourier-Mukai transform $\Phi_P(\mathcal{O}_X)$ is a $\text{gv}(X)$-syzygy sheaf, of rank $\chi(\omega_X)$; then apply the Evans-Griffith Syzygy Theorem.
(2) Inequalities for Hodge numbers; regularity.

**Theorem.** (Lazarsfeld – P., 2010) Too long to state, but point is that if $X$ has no irregular fibrations, then
\[ h^{p,0}(X) \geq f(q(X)). \]

(Generalization of various results of Castelnuovo-de Franchis, Catanese, Green-Lazarsfeld. Extended in some cases to arbitrary $h^{p,q}(X)$ by Lombardi.)

Same method gives: if $k$ is the dimension of the general fiber of the Albanese map, then:
\[ \bigoplus_{i=0}^{n} H^i(X, \omega_X) \text{ is } k\text{-regular over } E = \wedge^1 H^1(X, \mathcal{O}_X). \]

**Idea:** Use intersection theory and BGG correspondence for the globalized derivative complex governing the deformation theory of $H^i(X, L)$ with $L \in \text{Pic}^0(X)$. 

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**Ueno’s Conjecture.** Recently Chen and Hacon have proved part of a fundamental conjecture of Ueno regarding varieties with $\pi_1(X) = 0$.

**Theorem.** (Chen-Hacon, 2011) Let $X$ be a smooth projective variety with $\pi_1(X) = 0$, and $a: X \to A$ its Albanese map. Denote by $F$ the general fiber of $a$. Then:

1. $a$ is surjective and has connected fibers.
2. $\pi_1(F) = 0$.
3. More generally, the $C_{n,m}$ conjecture holds over a base of maximal Albanese dimension, i.e. $\pi_1(X) \rightarrow \pi_1(F) \oplus \pi_1(Y)$ if $f: X \to Y$ is a fiber space with $Y$ of maximal Albanese dimension.

**Idea:** Naively, one encodes $P_m(F)$ in $\bar{\pi}^*(\mathcal{O}_m \mathcal{X})$. More sophisticated approach is to look at $V_m = \bar{\pi}^*(\mathcal{O}_m \mathcal{X}) \oplus H^0(X, \mathcal{O}_X) + \cdots$. Roughly speaking, apply generic vanishing to $V_m$ (much more complicated in reality...)

**Holomorphic one-forms on varieties of general type.** More recent developments have led to a solution to the following conjecture of Hacon-Kovács and Luo-Zhang, partially due to Carrell as well.

**Theorem.** (P. – Schnell, ’12) If $X$ is of general type, then there are no nowhere vanishing holomorphic one-forms on $X$. 

Ueno’s Conjecture. Recently Chen and Hacon have proved part of a fundamental conjecture of Ueno regarding varieties with $\kappa(X) = 0$.

Theorem. (Chen-Hacon, ’11) Let $X$ be a smooth projective variety with $\kappa(X) = 0$, and $a : X \to A$ its Albanese map. Denote by $F$ the general fiber of $a$. Then:

(0) (Kawamata, ’80) $a$ is surjective and has connected fibers.

(1) $\kappa(F) = 0$.

(1’) More generally, the $C_{n,m}$ conjecture holds over a base of maximal Albanese dimension, i.e.

$$\kappa(X) \geq \kappa(F) + \kappa(Y)$$

if $f : X \to Y$ is a fiber space with $Y$ of maximal Albanese dimension.

Idea: Naively, one encodes $P_m(F)$ in the rank of $a_*(\omega_X^m)$. More sophisticated approach is to look at

$$V_m := a_*(\omega_X^m \otimes \mathcal{I}(\| (m - 1)K_X + a^*(\epsilon H) \|))$$

where $H$ is an ample line bundle on $A$. Roughly speaking, apply generic vanishing to $V_m$ (much more complicated in reality...)
For instance: let $a : X \to A = \text{Alb}(X)$ be the Albanese map of $X$. By the Decomposition Theorem, one has

$$a_* \mathbb{Q}_X[n] \simeq \bigoplus_i E_i[-i],$$

with $E_i$ (topological) perverse sheaves. Using the Riemann-Hilbert correspondence and M. Saito’s theory, we have a correspondence

$$E_i \leftrightarrow (\mathcal{M}_i, F),$$

where $(\mathcal{M}_i, F)$ is a filtered (regular, holonomic) $\mathcal{D}_A$-module. For such an $(\mathcal{M}, F)$, the associated graded object is

$$\text{gr}_F^\bullet \mathcal{M} = \bigoplus_k \text{gr}_F^k \mathcal{M},$$

which can be seen as a graded module over $\text{Sym}^\bullet T_A$ or, forgetting the grading, a coherent sheaf on the cotangent bundle $T^*A \simeq A \times V$, where $V = H^0(X, \Omega_X^1)$.
Theorem. For each Hodge $\mathcal{D}$-module $(\mathcal{M}, F)$ and each $k$, the sheaf $\text{gr}^k_F \mathcal{M}$ satisfies the Generic Vanishing package.

Example: $R^ia_*\omega_X$ arise in this fashion, and this case recovers the previous GV-theorems of Green-Lazarsfeld and Hacon.

However, this can be used in new situations. For instance, it leads to a Nakano-type generic vanishing theorem (which can be shown to be optimal):

Theorem. One has

$$\text{codim } V^q(\Omega^p_X) \geq |p + q - n| - \delta(a),$$

where $\delta(a)$ is the defect of semismallness of the Albanese map, i.e.

$$\delta(a) := \max_{l \in \mathbb{N}} (2l - \dim X + \dim A_l),$$

with $A_l$ the locus of points over which the fibers have dimension at least $l$. 
**Holomorphic one-forms on varieties of general type.** More recent developments have led to a solution to the following conjecture of Hacon-Kovács and Luo-Zhang, partially due to Carrell as well.

**Theorem.** (P. – Schnell, ’12) *If $X$ is of general type, then there are no nowhere vanishing holomorphic one-forms on $X$.***

One interesting corollary of this, combined with Kollár’s $C^+_{n,m}$ for morphisms with fibers of general type, is the following (special cases previously due to Kovács, Viehweg-Zuo):

**Corollary.** *If $f : X \to A$ is a smooth family of varieties of general type over an abelian base, then $f$ is birationally isotrivial.*

**Idea:** Use the Generic Vanishing package for Hodge $\mathcal{D}$-modules. Main point is a stronger statement about the total associated graded object $\text{gr}^\bullet_F \mathcal{M}$, or even better about the Laumon-Rothstein version of the Fourier-Mukai transform applied to $\mathcal{O}_X$ as a $\mathcal{D}$-module.
HAPPY BIRTHDAY ROB!