Permutation models, topological groups and Ramsey actions

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Abstract

Permutation models in which the axiom of choice fails but the Boolean prime ideal theorem holds correspond with a particular class of topological group. This result, due to Andreas Blass [2], is proved using combinatorial group theory, amongst other methods. In this talk I will illustrate the proof.

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1 The ZFA construction

I assume the audience is familiar with permutation models, but it won’t hurt to refresh your memory and make my notation and conventions clear. ZFA is the theory ZF modified to allow for the existence of atoms, empty objects which are not sets but which can be elements of sets. I will always assume that the collection $A$ of atoms forms a set, which is infinite.

The ZFA universe is $V = \bigcup_{\alpha\in \text{Ord}} \mathcal{P}^\alpha(A)$, where

$$
\mathcal{P}^0(A) = A, \quad \mathcal{P}^{\alpha+1}(A) = \mathcal{P}(\mathcal{P}^\alpha(A)), \quad \mathcal{P}^\lambda(A) = \bigcup_{\alpha<\lambda} \mathcal{P}^\alpha(A)
$$

for ordinals $\alpha, \lambda$ with $\lambda$ a limit ordinal.

If $G$ is a group of permutations of $A$ then a normal filter $\mathcal{F}$ on $G$ is a collection of subgroups of $G$ which is closed under . . .

- . . . supergroups: if $K \in \mathcal{F}$ and $K \leq H \leq G$ then $H \in \mathcal{F}$;
- . . . pairwise intersections: if $K, H \in \mathcal{F}$ then $K \cap H \in \mathcal{F}$;
- . . . conjugation: if $H \in \mathcal{F}$ and $\pi \in G$ then $\pi H \pi^{-1} \in \mathcal{F}$

Given a set $x$, its stabiliser, denoted $\text{stab}_G(x)$ is defined by

$$
\text{stab}_G(x) = \{ \pi \in G : \pi(x) = x \}
$$

A set $x$ is $\mathcal{F}$-symmetric if $\text{stab}_G(x) \in \mathcal{F}$. It so happens that given any group $G \leq \text{Aut}(A)$ and normal filter $\mathcal{F}$ on $G$, the collection of hereditarily $\mathcal{F}$-symmetric sets forms a (class) model of ZFA sitting inside $V$, called a permutation model. This model is denoted $\mathbb{M}(A, G, \mathcal{F})$, and will usually be abbreviated to $\mathbb{M}$ if $A, G$ and $\mathcal{F}$ are obvious from context.

2 Topological groups

A topological group is a set $G$ which is a group with operation $\cdot$ and a space with topology $\mathcal{O}$ such that the group operations

$$
(-) \cdot (-) : G \times G \to G, \quad (-)^{-1} : G \to G
$$

are continuous with respect to $\mathcal{O}$. In particular, all maps of the form $x \mapsto g \cdot x$ for $g, x \in G$ are homeomorphisms.

There is a kind of duality between topological groups and groups-with-a-normal-filter. Given a topology $\mathcal{O}$ on a topological group $G$, define

$$
\mathcal{O}^\dagger = \{ H \leq G : H \in \mathcal{O} \} = \{ \mathcal{O}\text{-open subgroups of } G \}
$$

Lemma 2.1 $\mathcal{O}^\dagger$ is a normal filter on $G$. $\square$

We can also obtain topologies from normal filters: given a normal filter $\mathcal{F}$ on $G$, define

$$
\mathcal{F}^\dagger = \{ U \subseteq G : (\forall g \in U)(\exists H \in \mathcal{F})(gH \subseteq U) \}
$$

That is, $U \in \mathcal{F}^\dagger$ if and only if every translation of $U$ onto the identity contains a subgroup in $\mathcal{F}$.
Lemma 2.2 \( \mathcal{F}^\dagger \) is a topology on \( G \) with respect to which the group operations are continuous. Furthermore, \( \mathcal{F}^{\dagger\dagger} = \mathcal{F} \). □

From now on, all the permutation models we consider will be of the form \( \mathcal{M}(A,G,\mathcal{O}^\dagger) \) where \( G \) is a topological group of permutations of \( A \) with topology \( \mathcal{O} \).

Naturally, certain properties of \( G \) will correspond with certain properties of \( \mathcal{M} \). The goal of the rest of this talk is to explore a couple of them.

3 The axiom of choice

Given a set \( x \) in \( \mathcal{V} \) and a group \( G \) of permutations of \( A \), the fixator of \( x \), denoted \( \text{fix}_G(x) \), is defined by

\[
\text{fix}_G(x) = \{ \pi \in G : (\forall y \in x)(\pi(y) = y) \} = \bigcap_{y \in x} \text{stab}_G(y)
\]

Lemma 3.1 A set \( x \) in \( \mathcal{M} \) has a wellorder in \( \mathcal{M} \) if and only if \( \text{fix}_G(x) \in \mathcal{F} \).

Proof By the axiom of choice in \( \mathcal{V} \) there exists an ordinal \( \alpha \) and a bijection \( f : x \to \alpha \) in \( \mathcal{V} \). Since \( \alpha \) is pure, each \( f(y) \) is pure, so for any \( \pi \in G \) and \( y \in x \), \( \pi(f(y)) = f(y) \). Thus \( \pi(f(y)) = f(y) \) for all \( y \in x \). So

\[
\pi \in \text{stab}_G(f) \iff \pi(f) = f \iff (\forall y \in x)(\pi(y) = y) \iff \pi \in \text{fix}_G(x)
\]

So \( \text{stab}_G(f) = \text{fix}_G(x) \). It follows that \( f \) lies in \( \mathcal{M} \) if and only if \( \text{fix}_G(x) \in \mathcal{F} \). □

In the case when \( G \) is a topological group and \( \mathcal{F} = \mathcal{O}^\dagger \), we can classify permutation models in which the axiom of choice holds with ease:

Proposition 3.2 Suppose \( G \) is a topological group with topology \( \mathcal{O} \). The axiom of choice holds in \( \mathcal{M}(A,G,\mathcal{O}^\dagger) \) if and only if \( G \) is discrete.

Proof First note that \( G \) is discrete if and only if \( \{1\} \in \mathcal{O}^\dagger \). The direction \( \Rightarrow \) is clear. For \( \Leftarrow \), note that if \( g \in G \) then \( \{g\} = g(\{1\}) \in \mathcal{O}^\dagger \).

Now if the axiom of choice holds then in particular \( \{1\} = \text{fix}_G(A) \in \mathcal{O}^\dagger \). Conversely, if \( \{1\} \in \mathcal{O}^\dagger \) then each \( \text{fix}_G(x) \) lies in \( \mathcal{O}^\dagger \) by upwards-closure. □

This prove illustrates the following correspondence:

| permutation models satisfying AC | filters containing \( \{1\} \) | discrete groups |

To translate between properties of the permutation model and properties of the group, we had to go via properties of the filter. A similar pattern will be repeated (in a much more long-winded fashion) in the more interesting correspondence below.

2
4 The Boolean prime ideal theorem

The Boolean prime ideal theorem (BPIT) is the assertion that every ideal in a Boolean algebra can be extended to a prime ideal. It is in fact equivalent to the apparently weaker assertion that every Boolean algebra contains a prime ideal.

This form of BPIT isn’t particularly useful for our purposes, but we have other equivalent assertions at our fingertips. Namely, the compactness theorem for propositional logic (CT) and the almost maximal ideal theorem (AMIT; see [4]). The latter is the assertion that every Boolean algebra \( B \) has an almost maximal ideal, i.e. a prime ideal \( I \) such that

\[
I = \widehat{I} = \{ x \in B : (\forall y \in B)(x \vee y = 1 \rightarrow (\exists z \in I)(y \vee z = 1)) \}
\]

This can be put more succinctly. Define the dual \( I^* \) of an ideal \( I \) by

\[
I^* = \{ y \in B : (\exists z \in I)(y \vee z = 1) \}
\]

Then \( \widehat{I} = \{ x \in B : (\forall y \in B)(x \vee y = 1 \rightarrow y \in I^*) \} \). We will revisit the dual later.

Permutation models in which BPIT holds correspond with an interesting class of topological groups, which in turn correspond with an interesting class of filters. Namely

| permutation models satisfying BPIT | Ramsey filters | extremely amenable groups with small open subgroups |

These terms will be defined in the next couple of sections, and a sketch proof will follow.

5 Extremely amenable groups

Let \( G \) be a group with topology \( \mathcal{O} \). We say \( G \) is extremely amenable if

- \( G \) is Hausdorff; and
- whenever \( G \) acts continuously on a compact Hausdorff space \( X \), the action has a fixed point (i.e. \( x^* \in X \) such that \( g \cdot x^* = x^* \) for all \( g \in G \)).

The trivial group \( \{1\} \) is clearly extremely amenable. The existence of nontrivial examples isn’t an easy result, but they do exist. Examples of nontrivial extremely amenable groups include:

- \( \text{Aut}(\mathbb{Q},<) \), the group of order-preserving permutations of the rationals;
- \( \text{Homeo}_+(\mathbb{R}) \), the group of orientation-preserving homeomorphisms of the unit interval.

The former is given the product topology from \( \mathbb{Q}^\mathbb{Q} \) inherited from the discrete topology on \( \mathbb{Q} \), and the latter is given the product topology from \([0,1]^{[0,1]}\) inherited from the discrete topology on \([0,1] \). For proofs, see [6].

These examples satisfy the additional property of having small open subgroups, i.e. any neighbourhood of 1 contains an open subgroup. This is, in fact, equivalent to the assertion that \( \mathcal{O} \subseteq \mathcal{O}^{\uparrow} \).
6 Ramsey filters

An action of a group \(G\) on a set \(X\) is a Ramsey action if it satisfies the following Ramsey property: given any 2-colouring of \(X\) and any finite subset \(F \subseteq X\), there exists \(g \in G\) such that \(g \cdot F\) is monochromatic. This is in fact equivalent to a much weaker assertion:

**Lemma 6.1** An action of a group \(G\) on a set \(X\) is a Ramsey action if and only if for each finite \(F \subseteq X\) there exists a finite \(Y \subseteq X\) such that for each \(k\)-colouring of \(Y\), there exists \(g \in G\) such that \(g \cdot F \subseteq Y\) and \(g \cdot F\) is monochromatic. \(\square\)

A subgroup \(H \leq G\) is a Ramsey subgroup of \(G\) if the natural left-action of \(G\) on the set of cosets of \(H\) in \(G\) is a Ramsey action. Finally, a filter \(\mathcal{F}\) on \(G\) is a Ramsey filter if there exists \(H \in \mathcal{F}\) such that every subgroup of \(H\) is a Ramsey subgroup of \(H\). We then say \(\mathcal{F}\) is a Ramsey filter as witnessed by \(H\).

7 The correspondence

The correspondence, due to Blass [2], comes in two halves, namely Theorems 7.1 and 7.2. I will not prove 7.1, and I will sketch the proof of 7.2.

**Theorem 7.1** Let \(G\) be a Hausdorff topological group with topology \(\mathcal{O}\) and suppose \(G\) has small open subgroups. Then \(G\) is nontrivial and extremely amenable if and only if \(\mathcal{O}^\dagger\) is a Ramsey filter as witnessed by \(G\) and \(\{1\} \notin \mathcal{O}^\dagger\). \(\square\)

**Theorem 7.2** Let \(G\) be a Hausdorff topological group of permutations of \(A\) with topology \(\mathcal{O}\). Suppose that \(G\) has small open subgroups, that each \(a \in A\) is \(\mathcal{O}^\dagger\)-symmetric, and that each \(H \in \mathcal{F}\) appears as a stabiliser of some set in \(M(A,G,\mathcal{O}^\dagger)\).

Then \(M \models \text{BPIT}\) if and only if \(\mathcal{O}^\dagger\) is a Ramsey filter on \(G\). \(\square\)

8 Example: Mostowski’s ordered model

Mostowski’s ordered model is the permutation \(M = M(A,G,\mathcal{F})\), where

- \(A\) is countable and equipped with an ordering \(<_A\) such that \(\langle A,<_A\rangle \cong \langle \mathbb{Q},<_\mathbb{Q}\rangle\);
- \(G = \text{Aut}(\langle A,<_A\rangle)\) is the group of order-preserving permutations of \(A\);
- \(\mathcal{F}\) is the collection of subgroups of \(G\) containing as a subgroup a stabiliser of a finite subset of \(A\), i.e.
  \[ H \in \mathcal{F} \iff (\exists F \subseteq A \text{ finite})(\text{stab}_G(F) \leq H) \]

We already know that, when endowed with the appropriate topology \(\mathcal{O}\), \(G\) is a nontrivial extremely amenable group with small open subgroups. Specifically, \(\mathcal{O}\) is the (subspace topology of the) product topology inherited from the discrete topology on \(A\). We should probably prove the following proposition before moving on.
Proposition 8.1 With $F$ and $O$ as above, $F = O^\dagger$. □

Proof The product topology on $A^4$ is generated by unions of finite intersections of sets of the form $\pi_a^{-1}(\{b\})$ for $a, b \in A$. If $1 \in \pi_a^{-1}(\{b\})$ then $a = b$ and $\pi_a^{-1}(\{a\}) = \text{fix}_G(\{a\}) = \text{stab}_G(a)$. Given a finite subset $F \subseteq A$

$$\bigcap_{a \in F} \pi_a^{-1}(\{a\}) = \bigcap_{a \in F} \text{stab}_G(a) = \text{fix}_G(F)$$

Thus the open neighbourhoods of 1 are unions of sets of the form $\text{fix}_G(F)$ for $F \subseteq A$ finite. Hence $U \subseteq A^4$ with $1 \in U$ is open if and only if it contains some $\text{fix}_G(F)$ as a subset, and in particular $H \leq G$ is open if and only if $H \in \mathcal{F}$. □

Theorem 8.2 Mostowski’s ordered model satisfies the Boolean prime ideal theorem

Proof 1 We can prove this directly using the equivalence between BPIT and the consistency principle and proving that the consistency principle holds in $M$. □

Proof 2 It is known that $G$ is extremely amenable and has small open subgroups. It also satisfies the additional hypotheses of Theorem 7.2. By Theorem 7.1 $O^\dagger$ has the Ramsey property as witnessed by $G$. □

9 What next?

We’ve seen two examples now where we translate between properties of $M$ and properties of $G$ via properties of $O^\dagger$. But BPIT is not a very strong weakening of the axiom of choice; which properties of $G$ does the ordering principle correspond with? What about the axiom of choice for sets of pairs? To the best of my knowledge, these questions remain open.

References


