Categories of natural models of type theory
ASL Logic Colloquium 2016 (Leeds, UK)

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1. Review of background material

2. Algebraic description of homomorphisms

3. Functorial description of homomorphisms

4. Interpreting the syntax
Representable natural transformations

Let $\mathcal{C}$ be a category and let $\mathcal{U}, \tilde{\mathcal{U}} : \mathcal{C}^{\text{op}} \to \text{Set}$. Recall that a natural transformation $p : \mathcal{U} \to \mathcal{U}$ is *representable* if it satisfies one of the following equivalent conditions:
Representable natural transformations

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- For all $\Gamma \in \text{ob}(\mathcal{C})$ and all $A \in \mathcal{U}(\Gamma)$,

\[
\begin{array}{c}
\tilde{\mathcal{U}} \\
\downarrow p \\
\mathcal{U}
\end{array}
\]

\[
y(\Gamma) \quad A \quad \mathcal{U}
\]

The induced functor $\int_{\mathcal{C}} p : \int_{\mathcal{C}} \tilde{\mathcal{U}} \to \int_{\mathcal{C}} \mathcal{U}$ on categories of elements has a right adjoint.
Representable natural transformations

Let \( \mathcal{C} \) be a category and let \( \mathcal{U}, \tilde{\mathcal{U}} : \mathcal{C}^{\text{op}} \to \text{Set} \). Recall that a natural transformation \( p : \mathcal{U} \to \mathcal{U} \) is representable if it satisfies one of the following equivalent conditions:

- For all \( \Gamma \in \text{ob}(\mathcal{C}) \) and all \( A \in \mathcal{U}(\Gamma) \), there exist \( \Gamma \cdot A, p^\Gamma_A, q^\Gamma_A \) making the following diagram a pullback:

\[
\begin{array}{ccc}
\ y(\Gamma \cdot A) & \xrightarrow{q^\Gamma_A} & \tilde{\mathcal{U}} \\
\ | \downarrow & & \downarrow p \\
\ y(p^\Gamma_A) & & \ \mathcal{U}
\end{array}
\]
Representable natural transformations

Let $\mathcal{C}$ be a category and let $\mathcal{U}, \tilde{\mathcal{U}} : \mathcal{C}^{\text{op}} \to \text{Set}$. Recall that a natural transformation $p : \mathcal{U} \to \mathcal{U}$ is representable if it satisfies one of the following equivalent conditions:

- For all $\Gamma \in \text{ob}(\mathcal{C})$ and all $A \in \mathcal{U}(\Gamma)$, there exist $\Gamma \cdot A, p_A^\Gamma, q_A^\Gamma$ making the following diagram a pullback:

\[
\begin{array}{ccc}
y(\Gamma \cdot A) & \xrightarrow{q_A^\Gamma} & \tilde{\mathcal{U}} \\
y(p_A^\Gamma) \downarrow & & \downarrow p \\
y(\Gamma) & \xrightarrow{A} & \mathcal{U}
\end{array}
\]

- The induced functor $\int \mathcal{C} p : \int \mathcal{C} \tilde{\mathcal{U}} \to \int \mathcal{C} \mathcal{U}$ on categories of elements has a right adjoint.
Definition of a natural model

Definition

A **natural model** is an octuple \( \mathcal{C} = (\mathcal{C}, \odot, \mathcal{U}, \tilde{\mathcal{U}}, p, p^*, \eta, \varepsilon) \) consisting of the following data:
Definition of a natural model

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A **natural model** is an octuple \( \mathcal{C} = (\mathcal{C}, \Diamond, \mathcal{U}, \tilde{\mathcal{U}}, p, p^*, \eta, \varepsilon) \) consisting of the following data:

- A base category \( \mathcal{C} \) with a terminal object \( \Diamond \);
Definition of a natural model

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- A base category \( \mathcal{C} \) with a terminal object \( \diamond \);
- Presheaves \( \mathcal{U}, \mathcal{\tilde{U}} : \mathcal{C}^{\text{op}} \to \text{Set} \);
Definition of a natural model

Definition

A **natural model** is an octuple \( \mathcal{C} = (\mathbb{C}, \diamond, \mathcal{U}, \tilde{\mathcal{U}}, \rho, \rho^*, \eta, \varepsilon) \) consisting of the following data:

- A base category \( \mathbb{C} \) with a terminal object \( \diamond \);
- Presheaves \( \mathcal{U}, \tilde{\mathcal{U}} : \mathbb{C}^{\text{op}} \to \text{Set} \);
- Functors

\[
\int_{\mathbb{C}} \tilde{\mathcal{U}} \xrightarrow{\rho} \int_{\mathbb{C}} \mathcal{U}
\]

\[
\int_{\mathbb{C}} \tilde{\mathcal{U}} \xleftarrow{\rho^*} \int_{\mathbb{C}} \mathcal{U}
\]

such that \( \rho \) commutes with the projection maps to \( \mathbb{C} \);
Definition of a natural model

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- A base category \( \mathcal{C} \) with a terminal object \( \diamond \);
- Presheaves \( \mathcal{U}, \tilde{\mathcal{U}} : \mathcal{C}^{\text{op}} \to \text{Set} \);
- Functors
  \[
  \int_{\mathcal{C}} \tilde{\mathcal{U}} \xrightarrow{p} \int_{\mathcal{C}} \mathcal{U} \xleftarrow{p^*}
  \]
  such that \( p \) commutes with the projection maps to \( \mathcal{C} \);
- Natural transformations
  \[
  \eta : \text{id} \to p^* \circ p \quad \text{and} \quad \varepsilon : p \circ p^* \to \text{id}
  \]
  forming the unit and counit, respectively, of an adjunction \( p \dashv p^* \).
Outline

We’ll follow the standard pattern for functorial semantics:
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- Show that the syntax for type theory on a given signature $\Sigma$ presents the *free* natural model $\mathcal{Z}$ on $\Sigma$;
We’ll follow the standard pattern for functorial semantics:

- Define the notion of *homomorphism* of natural models;
- Show that the syntax for type theory on a given signature $\Sigma$ presents the *free* natural model $T$ on $\Sigma$;
- $\rightsquigarrow$ An interpretation of $\Sigma$ in a natural model $\mathcal{C}$ is given by a homomorphism $T \rightarrow \mathcal{C}$. 
1. Review of background material
2. Algebraic description of homomorphisms
3. Functorial description of homomorphisms
4. Interpreting the syntax
Algebraic description of homomorphisms

Definition
Let \( \mathcal{C} = (C, \diamond, U, \tilde{U}, p, p^*, \eta, \varepsilon) \) and \( \mathcal{D} = (D, \bullet, V, \tilde{V}, q, q^*, \sigma, \tau) \) be natural models. A homomorphism from \( \mathcal{C} \) to \( \mathcal{D} \) is a triple \((F, \Phi, \tilde{\Phi})\) consisting of:
Algebraic description of homomorphisms

Definition
Let $\mathcal{C} = (\mathcal{C}, \diamond, U, \tilde{U}, p, p^*, \eta, \varepsilon)$ and $\mathcal{D} = (\mathcal{D}, \bullet, V, \tilde{V}, q, q^*, \sigma, \tau)$ be natural models. A homomorphism from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(F, \Phi, \tilde{\Phi})$ consisting of:

- A functor $F : \mathcal{C} \to \mathcal{D}$;
Algebraic description of homomorphisms

Definition
Let $\mathcal{C} = (C, \diamond, U, \tilde{U}, p, p^*, \eta, \varepsilon)$ and $\mathcal{D} = (D, \bullet, V, \tilde{V}, q, q^*, \sigma, \tau)$ be natural models. A **homomorphism** from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(F, \Phi, \tilde{\Phi})$ consisting of:

- A functor $F : C \to D$;
- Functors
  \[
  \Phi : \int_C U \to \int_D V \quad \text{and} \quad \tilde{\Phi} : \int_C \tilde{U} \to \int_D \tilde{V}
  \]

such that...
Algebraic description of homomorphisms

... the following diagrams commute (highlighted in red):

\[ \int_C \tilde{U} \xrightarrow{\tilde{\Phi}} \int_D \tilde{V} \]

\[ \begin{array}{c}
\int_C U \\
\downarrow p
\end{array} \xrightarrow{p^*} \begin{array}{c}
\int_D V \\
\downarrow q
\end{array} \]

\[ \begin{array}{c}
\int_C U \\
\downarrow \Phi
\end{array} \xrightarrow{\Phi} \begin{array}{c}
\int_D V \\
\downarrow F
\end{array} \]

Action on types respects context and substitution
Algebraic description of homomorphisms

... the following diagrams commute (highlighted in red):

\[
\begin{array}{ccc}
\int_C \tilde{U} & \xrightarrow{\tilde{\Phi}} & \int_D \tilde{V} \\
p & \quad & q \\
\downarrow & \quad & \downarrow \\
\int_C U & \xrightarrow{\Phi} & \int_D V \\
\end{array}
\]

\[
\begin{array}{cc}
\int_C \tilde{U} & \xrightarrow{\tilde{\Phi}} & \int_D \tilde{V} \\
\end{array}
\]

Action on terms respects context and substitution
Algebraic description of homomorphisms

... the following diagrams commute (highlighted in red):

\[ \int_C \tilde{U} \xrightarrow{\tilde{\Phi}} \int_D \tilde{V} \]

\[ \int_C U \xrightarrow{\Phi} \int_D V \]

Action on terms respects typing
Algebraic description of homomorphisms

... the following diagrams commute (highlighted in red):

Action on contexts and substitutions respects context extension
Algebraic description of homomorphisms

... and \( \Phi, \tilde{\Phi} \) respect the adjunctions \((p \dashv p^*, \eta, \varepsilon)\) and \((q \dashv q^*, \sigma, \tau)\), i.e.
Algebraic description of homomorphisms

... and $\Phi, \tilde{\Phi}$ respect the adjunctions $(p \dashv p^*, \eta, \varepsilon)$ and $(q \dashv q^*, \sigma, \tau)$, i.e.

\[
\begin{array}{ccc}
\int_C U & \xrightarrow{id} & \int_C U \\
\Phi & \downarrow \varepsilon & \Phi \\
\int_D V & \xrightarrow{id} & \int_D V \\
q \circ q^* & \downarrow \sigma & \end{array}
\]

- **Counit.** $\Phi \cdot \varepsilon = \tau \cdot \Phi \leadsto Fp_A^\Gamma = p_{FA}^{F\Gamma} : F\Gamma \cdot FA \to F\Gamma$
Algebraic description of homomorphisms

... and $\Phi, \tilde{\Phi}$ respect the adjunctions $(\rho \vdash \rho^*, \eta, \epsilon)$ and $(q \vdash q^*, \sigma, \tau)$, i.e.

\[
\begin{align*}
\int_C U & \xrightarrow{\Phi} \int_C U \\
\Phi \downarrow & \quad \downarrow \epsilon \\
\int_D V & \xleftarrow{q \circ q^*} \quad \quad \int_D V
\end{align*}
\]

\[
\begin{align*}
\int_C \tilde{U} & \xrightarrow{\tilde{\Phi}} \int_C \tilde{U} \\
\tilde{\Phi} \downarrow & \quad \downarrow \eta \\
\int_D \tilde{V} & \xleftarrow{q^* \circ q} \quad \quad \int_D \tilde{V}
\end{align*}
\]

- **Counit.** $\Phi \cdot \epsilon = \tau \cdot \Phi \quad \sim \quad FP^\Gamma_A = p_{FA}^{\Gamma} : F\Gamma \cdot FA \to F\Gamma$
- **Unit.** $\tilde{\Phi} \cdot \eta = \sigma \cdot \tilde{\Phi} \quad \sim \quad F\langle \text{id}_\Gamma, q_A^\Gamma \rangle = \langle \text{id}_{FA}^\Gamma, q_{FA}^{\Gamma} \rangle : F\Gamma \to F\Gamma \cdot FA$
Algebraic description of homomorphisms

... and $\Phi, \tilde{\Phi}$ respect the adjunctions $(p \dashv p^*, \eta, \varepsilon)$ and $(q \dashv q^*, \sigma, \tau)$, i.e.

\[
\begin{align*}
\int C U & \xrightarrow{p \circ p^*} \int C U \\
\Phi & \downarrow \text{id} \\
\int D V & \xrightarrow{q \circ q^*} \int D V
\end{align*}
\]

\[
\begin{align*}
\int \tilde{C} \tilde{U} & \xrightarrow{id} \int \tilde{C} \tilde{U} \\
\tilde{\Phi} & \downarrow \text{id} \\
\int \tilde{D} \tilde{V} & \xrightarrow{id} \int \tilde{D} \tilde{V}
\end{align*}
\]

- **Counit.** $\Phi \cdot \varepsilon = \tau \cdot \Phi \quad \leadsto \quad Fp^\Gamma_A = p^F\Gamma_{FA} : F\Gamma \cdot FA \to F\Gamma$

- **Unit.** $\tilde{\Phi} \cdot \eta = \sigma \cdot \tilde{\Phi} \quad \leadsto \quad F\langle \text{id}_\Gamma, q^\Gamma_A \rangle = \langle \text{id}_F\Gamma, q^F\Gamma_{FA} \rangle : F\Gamma \to F\Gamma \cdot FA$

... and $F(\Diamond) = \bullet$. 
Category of natural models

Theorem
There is a category $\text{NM}$, where:

- The objects of $\text{NM}$ are natural models;
- The morphisms of $\text{NM}$ are homomorphisms;
- The identity morphism on a natural model $\mathcal{C}$ is $(\text{id}_\mathcal{C}, \text{id}_\mathcal{U}, \text{id}_\mathcal{\tilde{U}})$;
- Composition is given componentwise:

$$(G, \psi, \tilde{\psi}) \circ (F, \phi, \tilde{\phi}) = (G \circ F, \psi \circ \phi, \tilde{\psi} \circ \tilde{\phi})$$
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\[(G, \Psi, \tilde{\Psi}) \circ (F, \Phi, \tilde{\Phi}) = (G \circ F, \Psi \circ \Phi, \tilde{\Psi} \circ \tilde{\Phi})\]

Since homomorphisms are defined diagramatically, this is extremely simple to prove.
1. Review of background material
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Remark on Kan extension

Any functor $F : \mathcal{C} \to \mathcal{D}$ between small categories induces an adjunction $F_! \dashv F^*$ between presheaf categories

$$
\begin{array}{c}
\text{Set}^{\mathcal{C}^{\text{op}}} \\
\overset{F_!}{\Longrightarrow} \\
\text{Set}^{\mathcal{D}^{\text{op}}}
\end{array}
\xrightarrow{F^*}

$$

where

- $F^* = - \circ F$ is precomposition with $F$; and
- $F_!$ is left Kan extension along $F$. 
Remark on Kan extension

Any functor $F : \mathcal{C} \to \mathcal{D}$ between small categories induces an adjunction $F_! \dashv F^*$ between presheaf categories

$$
\begin{array}{ccc}
\mathcal{C} & \xymatrix{ & \mathcal{D} \ar[ll]_{F} \ar[rr]^{F^*} & & \mathcal{D}^\text{op} \ar[ll]_{F_!}} & \mathcal{D} \\
& \ar[u]^{y} & \ar[u]_{y} \\
\mathcal{C}^\text{op} & \xymatrix{ & \mathcal{D}^\text{op} \ar[ll]_{F_!} \ar[rr]^{F^*} & & \mathcal{D}^\text{op} \ar[ll]_{F} & \mathcal{D}^\text{op}} & \mathcal{D}^\text{op} \\
\end{array}
$$

where

- $F^* = - \circ F$ is precomposition with $F$; and
- $F_!$ is left Kan extension along $F$.

Moreover, $F_! \circ y \simeq y \circ F : \mathcal{C} \to \mathcal{D}^\text{op}$. 

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Categories of natural models of type theory
Functorial presentation of homomorphisms

Specifying a homomorphism \((F, \Phi, \tilde{\Phi}) : \mathcal{C} \to \mathcal{D}\) is equivalent to specifying:

\[ F(\cdot A) = F(\cdot) \cdot F(A) \quad \text{for all } \Gamma \in \text{ob}(\mathcal{C}); \]

where \(\cdot\) denotes the internal homomorphism.
Functorial presentation of homomorphisms

Specifying a homomorphism \((F, \Phi, \tilde{\Phi}) : \mathcal{C} \to \mathcal{D}\) is equivalent to specifying:

- A functor \(F : \mathcal{C} \to \mathcal{D}\);
Functorial presentation of homomorphisms

Specifying a homomorphism \((F, \Phi, \tilde{\Phi}) : \mathcal{C} \to \mathcal{D}\) is equivalent to specifying:

- A functor \(F : \mathcal{C} \to \mathcal{D}\);
- Natural transformations \(\varphi : F_!U \to \mathcal{V}\) and \(\tilde{\varphi} : F_!\tilde{U} \to \tilde{\mathcal{V}}\)
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Specifying a homomorphism \((F, \Phi, \tilde{\Phi}) : \mathcal{C} \to \mathcal{D}\) is equivalent to specifying:

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- Natural transformations \(\varphi : F_i U \to \mathcal{V}\) and \(\tilde{\varphi} : F_i \tilde{U} \to \tilde{\mathcal{V}}\)

such that

- \(F(\diamond) = \bullet\);
Functorial presentation of homomorphisms

Specifying a homomorphism \((F, \Phi, \tilde{\Phi}) : \mathcal{C} \to \mathcal{D}\) is equivalent to specifying:

- A functor \(F : \mathcal{C} \to \mathcal{D}\);
- Natural transformations \(\varphi : F_! U \to V\) and \(\tilde{\varphi} : F_! \tilde{U} \to \tilde{V}\) such that
- \(F(\diamond) = \bullet\);
- The diagram \(\begin{array}{ccc} F_! \tilde{U} & \xrightarrow{\tilde{\varphi}} & \tilde{V} \\ F_! p \downarrow & & \downarrow q \\ F_! U & \xrightarrow{\varphi} & V \end{array}\) commutes;
Functorial presentation of homomorphisms

Specifying a homomorphism \((F, \Phi, \tilde{\Phi}) : \mathcal{C} \to \mathcal{D}\) is equivalent to specifying:

- A functor \(F : \mathcal{C} \to \mathcal{D}\);
- Natural transformations \(\varphi : F_iU \to \mathcal{V}\) and \(\tilde{\varphi} : F_i\tilde{U} \to \tilde{\mathcal{V}}\)

such that

- \(F(\diamond) = \bullet\);
- The diagram \[
\begin{array}{ccc}
F_i\tilde{U} & \xrightarrow{\tilde{\varphi}} & \tilde{\mathcal{V}} \\
\downarrow p & & \downarrow q \\
F_iU & \xrightarrow{\varphi} & \mathcal{V}
\end{array}
\]
commutes;
- \(F(\Gamma \cdot A) = F\Gamma \cdot FA\) for all \(\Gamma \in \text{ob}(\mathcal{C})\); and
Functorial presentation of homomorphisms

Specifying a homomorphism \((F, \Phi, \tilde{\Phi}) : \mathcal{C} \to \mathcal{D}\) is equivalent to specifying:

- A functor \(F : \mathcal{C} \to \mathcal{D}\);
- Natural transformations \(\varphi : F_U \to V\) and \(\tilde{\varphi} : F_{\tilde{U}} \to \tilde{V}\)

such that

- \(F(\circ) = \bullet\);
- The diagram \(\begin{array}{ccc} F_{\tilde{U}} & \xrightarrow{\tilde{\varphi}} & \tilde{V} \\ _F \downarrow_p & & \downarrow_q \\ F_U & \xrightarrow{\varphi} & V \end{array}\) commutes;
- \(F(\Gamma \cdot A) = F\Gamma \cdot FA\) for all \(\Gamma \in \text{ob}(\mathcal{C})\); and
- The comparison morphisms \(c^\Gamma_A : F(\Gamma \cdot A) \to F\Gamma \cdot FA\) are identities.
Action on types and terms

We obtain an action of $\varphi$ on types and $\tilde{\varphi}$ on terms as follows.
Action on types and terms

We obtain an action of \( \varphi \) on types and \( \tilde{\varphi} \) on terms as follows.

- **Action on types.** \( A \in \mathcal{U}(\Gamma) \leadsto FA \in \mathcal{V}(F\Gamma) \) via
Action on types and terms

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  $\ y(\Gamma) \xrightarrow{A} \mathcal{U} \ $
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\[
y(\Gamma) \xrightarrow{A} \mathcal{U} \quad \quad \quad \quad \quad F_!y(\Gamma) \xrightarrow{F_!A} F_!\mathcal{U}
\]
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- **Action on types.** $A \in \mathcal{U}(\Gamma) \leadsto FA \in \mathcal{V}(F\Gamma)$ via

$$y(\Gamma) \xrightarrow{A} \mathcal{U} \quad F_{!}y(\Gamma) \xrightarrow{F_{!}A} F_{!}\mathcal{U} \xrightarrow{\varphi} \mathcal{V}$$
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\[
\begin{align*}
  y(\Gamma) & \xrightarrow{A} \mathcal{U} \\
  F_!y(\Gamma) & \xrightarrow{F_!A} F_!\mathcal{U} \xrightarrow{\varphi} \mathcal{V} \\
  & \xrightarrow{\sim} \mathcal{V}
\end{align*}
\]
Action on types and terms

We obtain an action of $\varphi$ on types and $\tilde{\varphi}$ on terms as follows.

- **Action on types.** $A \in \mathcal{U}(\Gamma) \leadsto FA \in \mathcal{V}(F\Gamma)$ via

  \[ y(\Gamma) \xrightarrow{A} \mathcal{U} \quad \xrightarrow{\sim} \quad F_{!}y(\Gamma) \xrightarrow{F_{!}A} F_{!}\mathcal{U} \xrightarrow{\varphi} \mathcal{V} \]

  \[ \xrightarrow{\sim} \quad y(F\Gamma) \xrightarrow{FA} \]
We obtain an action of $\varphi$ on types and $\tilde{\varphi}$ on terms as follows.

- **Action on types.** $A \in \mathcal{U}(\Gamma) \rightsquigarrow FA \in \mathcal{V}(F\Gamma)$ via

  \[
  y(\Gamma) \xrightarrow{A} \mathcal{U} \quad \text{and} \quad F_{\downarrow} y(\Gamma) \xrightarrow{F_{\downarrow}A} F_{\downarrow} \mathcal{U} \xrightarrow{\varphi} \mathcal{V}
  \]

  \[
  \sim \quad \uparrow_{\mathcal{R}} \quad \sim \quad \to \quad FA
  \]

- **Action on terms.** $a \in \tilde{\mathcal{U}}(\Gamma) \rightsquigarrow Fa \in \tilde{\mathcal{V}}(F\Gamma)$ via

  \[
  y(\Gamma) \xrightarrow{a} \tilde{\mathcal{U}} \quad \text{and} \quad F_{\downarrow} y(\Gamma) \xrightarrow{F_{\downarrow}a} F_{\downarrow} \tilde{\mathcal{U}} \xrightarrow{\tilde{\varphi}} \tilde{\mathcal{V}}
  \]

  \[
  \sim \quad \uparrow_{\mathcal{R}} \quad \sim \quad \to \quad Fa
  \]
Where the comparison morphisms $c^\Gamma_A$ come from

Set-up: $A$ type in context $\Gamma$
Where the comparison morphisms $c^\Gamma_A$ come from

\[
y(\Gamma \cdot A) \quad \xrightarrow{q^\Gamma_A} \quad \tilde{U} \\
y(p^\Gamma_A) \quad \downarrow \quad \downarrow p \\
y(\Gamma) \quad \xrightarrow{A} \quad U
\]

Context extension of $\Gamma$ by $A$
Where the comparison morphisms $c_A^\Gamma$ come from

\[
\begin{align*}
F_1y(\Gamma \cdot A) & \xrightarrow{F_1q_A^\Gamma} F_1\tilde{U} \\
F_1y(p_A^\Gamma) & \downarrow \quad \quad \quad \downarrow F_1(p) \\
F_1y(\Gamma) & \xrightarrow{F_1A} F_1U \\
\end{align*}
\]

Apply $F_1$
Where the comparison morphisms $c^\Gamma_A$ come from

\[
\begin{align*}
  y(F(\Gamma \cdot A)) & \xrightarrow{\mathcal{R}} F_1 y(\Gamma \cdot A) & & \xrightarrow{F_1 q_A^\Gamma} & F_1 \tilde{U} \\
  y(F p_A^\Gamma) & & \downarrow F_1 y(p_A^\Gamma) & & \downarrow F_1(p) \\
  y(F \Gamma) & \xrightarrow{\mathcal{R}} F_1 y(\Gamma) & & \xrightarrow{F_1 A} & F_1 U
\end{align*}
\]

\[F_1 \circ y \simeq y(F-)\]
Where the comparison morphisms $c^\Gamma_A$ come from

\[
y(F(\Gamma \cdot A)) \xrightarrow{\cong} F!y(\Gamma \cdot A) \xrightarrow{F!q^\Gamma_A} F!\tilde{U} \xrightarrow{\varphi} \tilde{V}
\]

\[
y(Fp^\Gamma_A) \downarrow \quad F!y(p^\Gamma_A) \downarrow \quad F!(p) \downarrow \quad q
\]

\[
y(F\Gamma) \xrightarrow{\cong} F!y(\Gamma) \xrightarrow{F!A} F!U \xrightarrow{\varphi} V
\]

Paste square for $\varphi, \tilde{\varphi}$
Where the comparison morphisms $c^\Gamma_A$ come from

\[
\begin{align*}
y(F(\Gamma \cdot A)) & \quad \xrightarrow{Fq^\Gamma_A} \quad \tilde{\tilde{\nu}} \\
y(Fp^\Gamma_A) & \quad \downarrow \\
y(F\Gamma) & \quad \quad \xrightarrow{FA} \quad \nu
\end{align*}
\]

Action of $\varphi$ on types and $\tilde{\varphi}$ on terms
Where the comparison morphisms $c^\Gamma_A$ come from

Extend context $F\Gamma$ by $FA$
Where the comparison morphisms $c^\Gamma_A$ come from

Obtain $c^\Gamma_A : F(\Gamma \cdot A) \to F\Gamma \cdot FA$ as shown
Where the comparison morphisms $c^\Gamma_A$ come from

$c^\Gamma_A = \text{id} \Rightarrow Fp^\Gamma_A = p^{\Gamma A}_{FA} \quad \text{and} \quad Fq^\Gamma_A = q^{\Gamma A}_{FA}$. 
Proof of equivalence

The idea behind the proof of equivalence of defining homomorphisms ‘algebraically’ and defining them ‘functorially’ is as follows:
Proof of equivalence

The idea behind the proof of equivalence of defining homomorphisms ‘algebraically’ and defining them ‘functorially’ is as follows:

■ Write $\varphi' : \mathcal{U} \to F^*\mathcal{V}$ for the transpose of $\varphi : F_!\mathcal{U} \to \mathcal{V}$ with respect to the adjunction $F_! \dashv F^*$;
Proof of equivalence

The idea behind the proof of equivalence of defining homomorphisms ‘algebraically’ and defining them ‘functorially’ is as follows:

- Write \( \varphi' : \mathcal{U} \to F^* \mathcal{V} \) for the transpose of \( \varphi : F! \mathcal{U} \to \mathcal{V} \) with respect to the adjunction \( F! \dashv F^* \);
- Note there is an embedding \( \iota : \int_C F^* \mathcal{V} \to \int_D \mathcal{V} \);
Proof of equivalence

The idea behind the proof of equivalence of defining homomorphisms ‘algebraically’ and defining them ‘functorially’ is as follows:

- Write $\varphi' : \mathcal{U} \to F^* \mathcal{V}$ for the transpose of $\varphi : F_! \mathcal{U} \to \mathcal{V}$ with respect to the adjunction $F_! \dashv F^*$;
- Note there is an embedding $\iota : \int_{\mathcal{C}} F^* \mathcal{V} \hookrightarrow \int_{\mathcal{D}} \mathcal{V}$;
- Define $\Phi$ to be the composite $\int_{\mathcal{C}} \mathcal{U} \xrightarrow{\int_{\mathcal{C}} \varphi'} \int_{\mathcal{C}} F^* \mathcal{V} \xrightarrow{\iota} \int_{\mathcal{D}} \mathcal{V}$.
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- Define $\tilde{\Phi}$ likewise.

The assignment $(F, \varphi, \tilde{\varphi}) \mapsto (F, \Phi, \tilde{\Phi})$ is as required.
1. Review of background material

2. Algebraic description of homomorphisms

3. Functorial description of homomorphisms

4. Interpreting the syntax
Interpreting the syntax

We take a similar approach to that of S. Castellan, P. Clairambault, P. Dybjer (2015). The idea is as follows:
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- Work in a system of type theory with four kinds of judgements

\[ \Gamma = \Gamma' \vdash, \quad \Delta \vdash \gamma = \gamma' : \Gamma, \quad \Gamma \vdash A = A', \quad \Gamma \vdash a = a' : A \]

(We write \( \Gamma \vdash \) instead of \( \Gamma = \Gamma \vdash \), and so on.)
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■ From the syntax, build a natural model

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\mathcal{T} = (\mathbb{T}, [], \text{Ty}, \text{Tm}, \text{ty}, \text{ext}, \text{sub}, \text{proj})
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called the *term model* of the system.
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- From the syntax, build a natural model

\[ \mathfrak{T} = (\mathbb{P}, [], Ty, Tm, ty, ext, sub, proj) \]

called the term model of the system.

- \( \mathfrak{T} \) will (in a suitable sense) be the free natural model supporting the derivation rules for this system.
Example 1: basic syntax

With no rules for type formation, the term model is very simple:
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It is very easy to prove the following result.

**Theorem**

*This data defines a natural model $\mathcal{T}$, which is an initial object in $\text{NM}$.***
Example 2: adding a unit type

Consider the type theory obtained by adding a unit type 1, i.e. we add the following rules to our syntax:

\[
\begin{align*}
\vdash 1 \\
\vdash \ast : 1 \\
\vdash a : 1 \\
\vdash a = \ast : 1
\end{align*}
\]

The term model \( \mathcal{E} \) for this system is defined as follows:
Example 2: adding a unit type

Consider the type theory obtained by adding a unit type \( 1 \), i.e. we add the following rules to our syntax:

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\Gamma & \vdash \ast : 1 \\
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\end{align*}
\]

\[
\begin{align*}
\Gamma & \vdash a = \ast : 1 \\
\end{align*}
\]

The term model \( \mathcal{S} \) for this system is defined as follows:

- The objects of \( T \) are the empty context \([0] := []\) and finite strings of the form \([n] := [1 \cdot 1 \cdots 1]\) for \( n \geq 1\); 
  \[ n \text{ times} \]
Example 2: adding a unit type

Consider the type theory obtained by adding a unit type \( 1 \), i.e. we add the following rules to our syntax:

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\end{align*}
\]

The term model \( \mathcal{T} \) for this system is defined as follows:

- The objects of \( \mathcal{T} \) are the empty context \( [0] := [] \) and finite strings of the form \( [n] := [1 \cdot 1 \cdots 1] \) for \( n \geq 1 \);
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- There is a unique morphism \( \gamma_{n,m} : [n] \to [m] \) for all \( n, m \in \mathbb{N} \).
Example 2: adding a unit type

Consider the type theory obtained by adding a unit type $1$, i.e. we add the following rules to our syntax:

$$
\frac{}{\vdash 1} \quad \frac{}{\vdash \star : 1} \quad \frac{\vdash a : 1}{\vdash a = \star : 1}
$$

The term model $\mathcal{S}$ for this system is defined as follows:

- The objects of $\mathcal{T}$ are the empty context $[0] := []$ and finite strings of the form $[n] := [1 \cdot 1 \cdot \cdots 1]$ for $n \geq 1$; $n$ times
- There is a unique morphism $\gamma_{n,m} : [n] \to [m]$ for all $n, m \in \mathbb{N}$.
- $\text{Ty}([n]) = \{[1]\}$ and $\text{Ty}(\gamma_{n,m}) = \text{id}_{\{[1]\}}$;
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The term model \( \mathcal{T} \) for this system is defined as follows:

- The objects of \( \mathcal{T} \) are the empty context \( [0] := [] \) and finite strings of the form \( [n] := [1 \cdot 1 \cdot \cdots 1] \) for \( n \geq 1 \);
  
  \( n \) times

- There is a unique morphism \( \gamma_{n,m} : [n] \to [m] \) for all \( n, m \in \mathbb{N} \).

- \( \text{Ty}([n]) = \{[1]\} \) and \( \text{Ty}(\gamma_{n,m}) = \text{id}_{\{1\}} \);

- \( \text{Tm}([n]) = \{[\star]\} \) and \( \text{Tm}(\gamma_{n,m}) = \text{id}_{\{[\star]\}} \);
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\end{align*}
\]

The term model $\Xi$ for this system is defined as follows:

- The objects of $\Xi$ are the empty context $[0] := []$ and finite strings of the form $[n] := [1 \cdot 1 \cdot 1 \cdots 1]$ for $n \geq 1$;
- There is a unique morphism $\gamma_{n,m} : [n] \to [m]$ for all $n, m \in \mathbb{N}$.
- $\mathrm{Ty}([n]) = \{[1]\}$ and $\mathrm{Ty}(\gamma_{n,m}) = \mathrm{id}_{\{[1]\}}$;
- $\mathrm{Tm}([n]) = \{[\star]\}$ and $\mathrm{Tm}(\gamma_{n,m}) = \mathrm{id}_{\{[\star]\}}$;
- $\mathrm{ty}([n], [\star]) = ([n], [1])$ and $\mathrm{ext}([n], [1]) = ([n+1], [\star])$;
Example 2: adding a unit type

Consider the type theory obtained by adding a unit type $1$, i.e. we add the following rules to our syntax:

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\]

The term model $\mathcal{T}$ for this system is defined as follows:

- The objects of $\mathcal{T}$ are the empty context $[0] := []$ and finite strings of the form $[n] := [1 \cdot 1 \cdots 1]$ for $n \geq 1$; $n$ times
- There is a unique morphism $\gamma_{n,m} : [n] \to [m]$ for all $n, m \in \mathbb{N}$.
- $\text{Ty}([n]) = \{[1]\}$ and $\text{Ty}(\gamma_{n,m}) = \text{id}_{\{1\}}$;
- $\text{Tm}([n]) = \{[\star]\}$ and $\text{Tm}(\gamma_{n,m}) = \text{id}_{\{\star\}}$;
- $\text{ty}([n],[\star]) = ([n],[1])$ and $\text{ext}([n],[1]) = ([n+1],[\star])$;
- $\text{sub}([n],[\star]) = \gamma_{n,n+1}$ and $\text{proj}([n],[\star]) = \gamma_{n+1,n}$. 
Example 2: adding a unit type

A natural model $\mathcal{C}$ supports the unit type if there exist

$$1_{\mathcal{C}} \in U(\Diamond) \quad \text{and} \quad \star_{\mathcal{C}} \in \tilde{U}(\Diamond)$$

such that $p_{\Diamond}^{-1}(\{1_{\mathcal{C}}\}) = \{\star_{\mathcal{C}}\}.$
Example 2: adding a unit type

A natural model $\mathcal{C}$ supports the unit type if there exist

$$1_{\mathcal{C}} \in \mathcal{U}(\diamond) \quad \text{and} \quad \star_{\mathcal{C}} \in \tilde{\mathcal{U}}(\diamond)$$

such that $p_{\diamond}^{-1}(\{1_{\mathcal{C}}\}) = \{\star_{\mathcal{C}}\}$.

**Theorem**

*The data $\mathcal{T}$ on the previous slide defines a natural model, which is the free natural model supporting the unit type, i.e.*
Example 2: adding a unit type

A natural model $\mathcal{C}$ supports the unit type if there exist

$$1_{\mathcal{C}} \in U(\diamond) \quad \text{and} \quad \star_{\mathcal{C}} \in \tilde{U}(\diamond)$$

such that $p_\diamond^{-1}(\{1_{\mathcal{C}}\}) = \{\star_{\mathcal{C}}\}$.

**Theorem**

The data $\mathcal{Z}$ on the previous slide defines a natural model, which is the free natural model supporting the unit type, i.e. if $\mathcal{C}$ is any natural model supporting the unit type, and $1_{\mathcal{C}} \in U(\diamond)$ and $\star_{\mathcal{C}} \in \tilde{U}(\diamond)$ are as above, then there is a unique homomorphism

$$(F, \Phi, \tilde{\Phi}) : \mathcal{Z} \rightarrow \mathcal{C}$$

such that

$$F[1] = 1_\mathcal{C} \quad \text{and} \quad F[\star] = \star_\mathcal{C}$$
Future work on natural models

In the pipeline:
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- Uniform way of constructing the term model for a signature;
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- 2-category theoretic aspects;
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In the pipeline:

- Uniform way of constructing the term model for a signature;
- 2-category theoretic aspects;
- Lawvere duality for generalised algebraic theories;
- Investigation of the polynomial functor induced by $p : \tilde{U} \rightarrow U$. 
Acknowledgements and references

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Introduction and basic theory of natural models:

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Some analogous results for categories with families:

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Thank you for listening!