1. Dependent type theory

2. Natural models of type theory

3. Algebraic description of homomorphisms

4. Functorial description of homomorphisms

5. Interpreting the syntax
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Simple type theory

Simple type theory consists of...

- **Types:** $A, B, \ldots, A \times B, A \rightarrow B, \ldots, 0, 1, \mathbb{N}, \mathbb{Z}, \ldots$
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- **Equations**: $a = b : A, \text{pr}_1(\langle a, b \rangle) = a : A, \ldots$
- **Rules of inference**: For example,

\[
\begin{align*}
\frac{A \quad B}{A \times B} & \quad \frac{a : A \quad b : B}{\langle a, b \rangle : A \times B} & \quad \frac{c : A \times B}{\text{pr}_1(c) : A} \\
\end{align*}
\]
Dependent type theory

In *dependent* type theory, we also have

- **Contexts**: \( \Gamma, \Delta, \ldots \) (lists of typed variables)
Dependent type theory

In *dependent* type theory, we also have

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  e.g.

  $n : \mathbb{N} \vdash \text{List}(n)$  
  “In the context $n : \mathbb{N}$, $\text{List}(n)$ is a type.”
In *dependent* type theory, we also have

- **Contexts:** \( \Gamma, \Delta, \ldots \) (lists of typed variables)
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- **Type-formers:** If \( \Gamma, x : A \vdash B(x) \), then \( \Gamma \vdash \prod_{x : A} B(x) \).
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- **Type-formers:** If $\Gamma, x : A \vdash B(x)$, then $\Gamma \vdash \prod_{x : A} B(x)$.

- **Substitutions:** $\Delta \vdash \gamma : \Gamma \rightsquigarrow$ if $\Gamma \vdash a : A$ then $\Delta \vdash a\{\gamma\} : A\{\gamma\}$
Yet another approach to semantics?

There are many kinds of semantics for dependent type theory, including:

- Locally cartesian closed categories (Seely)
- Contextual categories, categories with attributes (Cartmell)
- Categories with families (Dybjer)
- Comprehension categories (Jacobs)
- Display map categories (Cambridge school)
- Weak factorisation systems (Awodey and Warren)
- Algebraic weak factorisation systems (Garner; Grandis & Tholen)
- Homotopical categories (Garner and van den Berg)
- B-systems, C-systems, and universes (Voevodsky)
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Representable natural transformations

Let $\mathcal{C}$ be a category and let $\mathcal{U}, \widetilde{\mathcal{U}} : \mathcal{C}^{\text{op}} \to \textbf{Set}$. A natural transformation $p : \mathcal{U} \to \mathcal{U}$ is \textit{representable} if, for all $\Gamma \in \text{ob}(\mathcal{C})$ and all $A \in \mathcal{U}(\Gamma)$,

\[
y(\Gamma) \quad \xrightarrow{A} \quad \mathcal{U}
\]

is a pullback.

\[
\begin{array}{ccc}
\widetilde{\mathcal{U}} & \xrightarrow{p} & \mathcal{U} \\
\downarrow & & \downarrow \\
y(\Gamma) & \xrightarrow{A} & \mathcal{U}
\end{array}
\]
Representable natural transformations

Let $\mathcal{C}$ be a category and let $\mathcal{U}, \tilde{\mathcal{U}} : \mathcal{C}^{\text{op}} \to \text{Set}$. A natural transformation $p : \tilde{\mathcal{U}} \to \mathcal{U}$ is representable if, for all $\Gamma \in \text{ob}(\mathcal{C})$ and all $A \in \mathcal{U}(\Gamma)$, there exist $\Gamma \cdot A, p^\Gamma_A, q^\Gamma_A$ making the following diagram a pullback:

\[
\begin{array}{ccc}
  y(\Gamma \cdot A) & \xrightarrow{q^\Gamma_A} & \tilde{\mathcal{U}} \\
y(p^\Gamma_A) \downarrow & & \downarrow p \\
y(\Gamma) & \xrightarrow{A} & \mathcal{U}
\end{array}
\]
Informal semantics

<table>
<thead>
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Clive Newstead (cnewstead@cmu.edu) Carnegie Mellon University
Categories of natural models of type theory — CT 2016
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\[ a : \Gamma \vdash a \{ \gamma \} : A \{ \gamma \} \tilde{\mu} \]

\[ y(\Gamma) \xrightarrow{A} \mathcal{U} \]
## Informal semantics

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| $\Gamma$ context  
$\Gamma \vdash A$  
$\Gamma \vdash a : A$  
$\Delta \vdash \gamma : \Gamma$ | $\Gamma \in \text{ob}(\mathbb{C})$  
$A \in \mathcal{U}(\Gamma)$ |

\[ y(\Gamma) \xrightarrow{\mathcal{U}} \mathcal{U} \]
\[ a \xrightarrow{\gamma} \mathcal{U} \]
\[ \gamma : \Delta \rightarrow \Gamma \text{ in } \mathbb{C} \]
Informal semantics

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</tr>
<tr>
<td>( \Delta \vdash \gamma : \Gamma )</td>
<td>( y(\Gamma) \xrightarrow{A} \mathcal{U} )</td>
</tr>
<tr>
<td>( \Delta \vdash a{\gamma} : A{\gamma} )</td>
<td>( \gamma : \Delta \to \Gamma ) in ( C )</td>
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\[
\begin{align*}
&y(\Delta) \xrightarrow{y(\gamma)} y(\Gamma) \xrightarrow{A} \mathcal{U} \\
&\mathcal{U} \xrightarrow{\tilde{U}} y(\Gamma) \xrightarrow{\gamma : \Delta \to \Gamma} \mathcal{U} \\
&\mathcal{U} \xrightarrow{\tilde{U}} y(\Gamma) \xrightarrow{\gamma : \Delta \to \Gamma} \mathcal{U} \\
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&\mathcal{U} \xrightarrow{\tilde{U}} y(\Gamma) \xrightarrow{\gamma : \Delta \to \Gamma} \mathcal{U}
\end{align*}
\]
Context extension $\iff$ representability

**Type theory:** Given the following:

$$
\Gamma \vdash A, \quad \Delta \vdash \gamma : \Gamma, \quad \Delta \vdash a : A\{\gamma\}
$$

There is a unique substitution $\Delta \vdash \langle \gamma, a \rangle : \Gamma \cdot A$, such that

$$
\Delta \vdash p^\Gamma_A \circ \langle \gamma, a \rangle = \gamma : \Gamma \quad \text{and} \quad \Gamma \vdash q^\Gamma_A\{\langle \gamma, a \rangle\} = a : A
$$
Context extension $\leftrightarrow$ representability

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\Gamma \vdash A, \quad \Delta \vdash \gamma : \Gamma, \quad \Delta \vdash a : A\{\gamma\}
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There is a unique substitution $\Delta \vdash \langle \gamma, a \rangle : \Gamma \cdot A$, such that

$$
\Delta \vdash p^\Gamma_A \circ \langle \gamma, a \rangle = \gamma : \Gamma \quad \text{and} \quad \Gamma \vdash q^\Gamma_A\{\langle \gamma, a \rangle\} = a : A
$$

**Representable natural transformation:**

\[
\begin{array}{ccc}
   y(\Delta) & \xrightarrow{a} & y(\langle \gamma, a \rangle) \\
   y(\langle \gamma, a \rangle) & \xrightarrow{q^\Gamma_A} & y(\Gamma \cdot A) \\
   y(\Gamma \cdot A) & \xrightarrow{p^\Gamma_A} & y(\Gamma) \\
   y(\Gamma) & \xrightarrow{A} & U \\
\end{array}
\]
Connection with categories with families

Theorem (Awodey, 2015)

*Specifying a category with families with base category $\mathcal{C}$ is equivalent to specifying a representable natural transformation between presheaves on $\mathcal{C}$.*
Connection with categories with families

Theorem (Awodey, 2015)

Specifying a category with families with base category $\mathcal{C}$ is equivalent to specifying a representable natural transformation between presheaves on $\mathcal{C}$.

A natural model is a representable natural transformation.
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*Specifying a category with families with base category $\mathcal{C}$ is equivalent to specifying a representable natural transformation between presheaves on $\mathcal{C}$.*

* A natural model is a representable natural transformation.

We seek an *essentially algebraic* definition.
Representability via categories of elements

**Lemma**

A natural transformation $p : \tilde{U} \to U$ is representable if and only if the induced functor on categories of elements $\int_C p : \int_C \tilde{U} \to \int_C U$ has a right adjoint $p^*$.
Representability via categories of elements

Lemma

A natural transformation $p : \tilde{\mathcal{U}} \to \mathcal{U}$ is representable if and only if the induced functor on categories of elements $\int_C p : \int_C \tilde{\mathcal{U}} \to \int_C \mathcal{U}$ has a right adjoint $p^*$.

- $p^*(\Gamma, A) = (\Gamma \cdot A, q^\Gamma_A)$
Representability via categories of elements

Lemma

A natural transformation \( p : \tilde{\mathcal{U}} \to \mathcal{U} \) is representable if and only if the induced functor on categories of elements \( \int_{\mathcal{C}} p : \int_{\mathcal{C}} \tilde{\mathcal{U}} \to \int_{\mathcal{C}} \mathcal{U} \) has a right adjoint \( p^* \).

- \( p^*(\Gamma, A) = (\Gamma \cdot A, q_A^\Gamma) \)
- \( \varepsilon_{(\Gamma, A)} = p_A^\Gamma : \Gamma \cdot A \to A \)
Representability via categories of elements

**Lemma**

A natural transformation $p : \tilde{U} \to U$ is representable if and only if the induced functor on categories of elements $\int_C p : \int_C \tilde{U} \to \int_C U$ has a right adjoint $p^*$.

- $p^*(\Gamma, A) = (\Gamma \cdot A, q^\Gamma_A)$
- $\varepsilon_{\Gamma, A} = p^\Gamma_A : \Gamma \cdot A \to A$
- $\gamma : \Delta \to \Gamma \rightsquigarrow p^*(\gamma)$ as follows:

$$
\begin{array}{c}
\Delta \cdot A \{\gamma\} \\
\downarrow p^A_{\Delta\{\gamma\}}
\end{array} \xrightarrow{p^*(\gamma)}
\begin{array}{c}
\Gamma \cdot A \\
\downarrow p^\Gamma_A
\end{array}
\quad \quad
\begin{array}{c}
\Delta \quad \gamma \\
\downarrow
\end{array} \xrightarrow{\gamma}
\begin{array}{c}
\Gamma
\end{array}
$$
Definition of a natural model

Definition

A **natural model** is an octuple \( \mathcal{C} = (C, \diamond, U, \tilde{U}, p, p^*, \eta, \varepsilon) \) consisting of the following data:
Definition of a natural model

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A **natural model** is an octuple \( \mathcal{C} = (C, \Diamond, U, \check{U}, p, p^*, \eta, \varepsilon) \) consisting of the following data:

- A base category \( C \) with a terminal object \( \Diamond \);
Definition of a natural model

**Definition**

A **natural model** is an octuple \( C = (\mathcal{C}, \diamond, \mathcal{U}, \tilde{\mathcal{U}}, p, p^\ast, \eta, \varepsilon) \) consisting of the following data:

- A base category \( \mathcal{C} \) with a terminal object \( \diamond \);
- Presheaves \( \mathcal{U}, \tilde{\mathcal{U}} : \mathcal{C}^{\text{op}} \to \text{Set} \);
Definition of a natural model

Definition

A natural model is an octuple \( \mathcal{C} = (\mathcal{C}, \diamond, \mathcal{U}, \tilde{\mathcal{U}}, p, p^*, \eta, \varepsilon) \) consisting of the following data:

- A base category \( \mathcal{C} \) with a terminal object \( \diamond \);
- Presheaves \( \mathcal{U}, \tilde{\mathcal{U}} : \mathcal{C}^{\text{op}} \to \text{Set} \);
- Functors

\[
\int_{\mathcal{C}} \tilde{\mathcal{U}} \quad \xrightarrow{p} \quad \int_{\mathcal{C}} \mathcal{U}
\]

such that \( p \) commutes with the projection maps to \( \mathcal{C} \);
Definition of a natural model

Definition
A **natural model** is an octuple \( \mathcal{C} = (\mathbb{C}, \diamond, \mathcal{U}, \tilde{\mathcal{U}}, p, p^*, \eta, \varepsilon) \) consisting of the following data:

- A base category \( \mathbb{C} \) with a terminal object \( \diamond \);
- Presheaves \( \mathcal{U}, \tilde{\mathcal{U}} : \mathbb{C}^{\text{op}} \to \text{Set} \);
- Functors

\[
\int_{\mathbb{C}} \tilde{\mathcal{U}} \leftrightarrow_{p} p^* \rightarrow \int_{\mathbb{C}} \mathcal{U}
\]

such that \( p \) commutes with the projection maps to \( \mathbb{C} \);
- Natural transformations

\[
\eta : \text{id} \to p^* \circ p \quad \text{and} \quad \varepsilon : p \circ p^* \to \text{id}
\]

forming the unit and counit, respectively, of an adjunction \( p \dashv p^* \).
We’ll follow the standard pattern for functorial semantics:
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- Define the notion of *homomorphism* of natural models;
Outline

We’ll follow the standard pattern for functorial semantics:

- Define the notion of *homomorphism* of natural models;
- Show that the syntax for type theory on a given signature $\Sigma$ presents the *free* natural model $\mathcal{Z}$ on $\Sigma$;
We’ll follow the standard pattern for functorial semantics:

- Define the notion of *homomorphism* of natural models;
- Show that the syntax for type theory on a given signature $\Sigma$ presents the *free* natural model $\mathcal{F}$ on $\Sigma$;
- An interpretation of $\Sigma$ in a natural model $\mathcal{C}$ is given by a homomorphism $\mathcal{F} \to \mathcal{C}$.
1. Dependent type theory
2. Natural models of type theory
3. Algebraic description of homomorphisms
4. Functorial description of homomorphisms
5. Interpreting the syntax
Algebraic description of homomorphisms

Definition
Let \( \mathcal{C} = (\mathbb{C}, \diamond, \mathcal{U}, \tilde{\mathcal{U}}, p, p^*, \eta, \varepsilon) \) and \( \mathcal{D} = (\mathbb{D}, \bullet, \mathcal{V}, \tilde{\mathcal{V}}, q, q^*, \sigma, \tau) \) be natural models. A **homomorphism** from \( \mathcal{C} \) to \( \mathcal{D} \) is a triple \((F, \Phi, \tilde{\Phi})\) consisting of:
Algebraic description of homomorphisms

**Definition**

Let $\mathcal{C} = (\mathbb{C}, \odot, \mathcal{U}, \tilde{\mathcal{U}}, p, p^*, \eta, \varepsilon)$ and $\mathcal{D} = (\mathbb{D}, \bullet, \mathcal{V}, \tilde{\mathcal{V}}, q, q^*, \sigma, \tau)$ be natural models. A **homomorphism** from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(F, \Phi, \tilde{\Phi})$ consisting of:

- A functor $F : \mathbb{C} \rightarrow \mathbb{D}$;
Algebraic description of homomorphisms

Definition
Let \( \mathcal{C} = (\mathcal{C}, \odot, \mathcal{U}, \tilde{\mathcal{U}}, \rho, \rho^*, \eta, \varepsilon) \) and \( \mathcal{D} = (\mathcal{D}, \bullet, \mathcal{V}, \tilde{\mathcal{V}}, q, q^*, \sigma, \tau) \) be natural models. A homomorphism from \( \mathcal{C} \) to \( \mathcal{D} \) is a triple \((F, \Phi, \tilde{\Phi})\) consisting of:

- A functor \( F : \mathcal{C} \rightarrow \mathcal{D} \);
- Functors

\[
\Phi : \int_{\mathcal{C}} \mathcal{U} \rightarrow \int_{\mathcal{D}} \mathcal{V} \quad \text{and} \quad \tilde{\Phi} : \int_{\mathcal{C}} \tilde{\mathcal{U}} \rightarrow \int_{\mathcal{D}} \tilde{\mathcal{V}}
\]

such that...
Algebraic description of homomorphisms

... the following diagrams commute (highlighted in red):

\[
\begin{array}{ccc}
\int_C \widetilde{U} & \xrightarrow{\Phi} & \int_D \widetilde{V} \\
p \downarrow & & q \downarrow \\
\int_C U & \xrightarrow{\Phi} & \int_D V \\
\end{array}
\]

Action on types respects context and substitution

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Categories of natural models of type theory — CT 2016
Algebraic description of homomorphisms

... the following diagrams commute (highlighted in red):

\[
\begin{array}{ccc}
\int_C \tilde{U} & \xrightarrow{\tilde{\Phi}} & \int_D \tilde{V} \\
p & \overset{p^*}{\downarrow} & \overset{q^*}{\downarrow} \\
\int_C U & \xrightarrow{\Phi} & \int_D V \\
\end{array}
\]

Action on terms respects context and substitution
Algebraic description of homomorphisms

... the following diagrams commute (highlighted in red):

\[
\begin{array}{ccc}
\int_C \tilde{U} & \xrightarrow{\tilde{\Phi}} & \int_D \tilde{V} \\
\downarrow p & & \downarrow q \\
\int_C U & \xrightarrow{\Phi} & \int_D V \\
\end{array}
\]

Action on terms respects typing
Algebraic description of homomorphisms

... the following diagrams commute (highlighted in red):

\[ \begin{array}{ccc}
\int C \tilde{U} & \xrightarrow{\tilde{\Phi}} & \int D \tilde{V} \\
\downarrow p & & \downarrow q \\
\int C U & \xrightarrow{\Phi} & \int D V \\
\end{array} \]

Action on contexts and substitutions respects context extension

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Categories of natural models of type theory — CT 2016
Algebraic description of homomorphisms

... and $\Phi, \tilde{\Phi}$ respect the adjunctions $(p \dashv p^*, \eta, \varepsilon)$ and $(q \dashv q^*, \sigma, \tau)$, i.e.
Algebraic description of homomorphisms

... and $\Phi, \tilde{\Phi}$ respect the adjunctions $(p \dashv p^*, \eta, \varepsilon)$ and $(q \dashv q^*, \sigma, \tau)$, i.e.

\[
\begin{array}{c}
\int_C U \xrightarrow{p \circ p^*} \int_C U \\
\Phi \downarrow \quad \text{id} \quad \downarrow \Phi \\
\int_D V \xleftarrow{q \circ q^*} \int_D V
\end{array}
\]

- **Counit.** $\Phi \cdot \varepsilon = \tau \cdot \Phi \quad \implies \quad FP_A^\Gamma = p^{F\Gamma}_{FA} : F\Gamma \cdot FA \to F\Gamma$
Algebraic description of homomorphisms

... and $\Phi, \tilde{\Phi}$ respect the adjunctions $(\rho \dashv \rho^*, \eta, \varepsilon)$ and $(q \dashv q^*, \sigma, \tau)$, i.e.

\[
\begin{align*}
&\int_C U \xrightarrow{p \circ p^*} \int_C U \\
&\downarrow \varepsilon & \downarrow \varepsilon \\
&\Phi \downarrow \text{id} & \Phi \\
&\int_D V \xrightarrow{q \circ q^*} \int_D V
\end{align*}
\]

\[
\begin{align*}
&\int_{\tilde{C}} \tilde{U} \xrightarrow{id} \int_{\tilde{C}} \tilde{U} \\
&\downarrow \eta & \downarrow \eta \\
&\tilde{\Phi} \downarrow \text{id} & \tilde{\Phi} \\
&\int_{\tilde{D}} \tilde{V} \xrightarrow{id \circ q^* \circ q} \int_{\tilde{D}} \tilde{V}
\end{align*}
\]

- **Counit.** $\Phi \cdot \varepsilon = \tau \cdot \Phi \implies Fp^\Gamma_A = p^{F\Gamma}_{FA} : F\Gamma \cdot FA \to F\Gamma$
- **Unit.** $\tilde{\Phi} \cdot \eta = \sigma \cdot \tilde{\Phi} \implies F\langle \text{id}_\Gamma, q^\Gamma_A \rangle = \langle \text{id}_{F\Gamma}, q^{F\Gamma}_{FA} \rangle : F\Gamma \to F\Gamma \cdot FA$
Algebraic description of homomorphisms

... and $\Phi, \tilde{\Phi}$ respect the adjunctions $(\rho \dashv p^*, \eta, \varepsilon)$ and $(q \dashv q^*, \sigma, \tau)$, i.e.

\[ \int_C U \xrightarrow{\rho \circ p^*} \int_C U \xrightarrow{\Phi} \int_D V \xrightarrow{\Phi} \int_D V \xrightarrow{q \circ q^*} \]

\[ \int_C U \xrightarrow{id} \int_C U \xrightarrow{\Phi} \int_D V \xrightarrow{id} \int_D V \xrightarrow{\Phi} \]

\[ \int_C \tilde{U} \xrightarrow{id} \int_C \tilde{U} \xrightarrow{\tilde{\Phi}} \int_D \tilde{V} \xrightarrow{id} \int_D \tilde{V} \xrightarrow{\tilde{\Phi}} \]

\[ \int_C \tilde{U} \xrightarrow{\eta} \int_C \tilde{U} \xrightarrow{\tilde{\Phi}} \int_D \tilde{V} \xrightarrow{\sigma} \int_D \tilde{V} \xrightarrow{q^* \circ q} \]

\[ \int_C \tilde{U} \xrightarrow{\eta} \int_C \tilde{U} \xrightarrow{\tilde{\Phi}} \int_D \tilde{V} \xrightarrow{\tau} \int_D \tilde{V} \xrightarrow{q^* \circ q} \]

- **Counit.** $\Phi \cdot \varepsilon = \tau \cdot \Phi$ $\leadsto$ $Fp_A^\Gamma = p_{FA}^{F\Gamma} : F\Gamma \cdot FA \to F\Gamma$

- **Unit.** $\tilde{\Phi} \cdot \eta = \sigma \cdot \tilde{\Phi}$ $\leadsto$ $F\langle \text{id}_\Gamma, q_A^\Gamma \rangle = \langle \text{id}_{F\Gamma}, q_{FA}^{F\Gamma} \rangle : F\Gamma \to F\Gamma \cdot FA$

... and $F(\diamond) = \bullet$. 
Category of natural models

Theorem
There is a category $\mathbf{NM}$, where:

- The objects of $\mathbf{NM}$ are natural models;
- The morphisms of $\mathbf{NM}$ are homomorphisms;
- The identity morphism on a natural model $\mathcal{C}$ is $(\text{id}_\mathcal{C}, \text{id}_\int \mathcal{U}, \text{id}_\int \tilde{\mathcal{U}})$;
- Composition is given componentwise:

$$ (G, \psi, \tilde{\psi}) \circ (F, \phi, \tilde{\phi}) = (G \circ F, \psi \circ \phi, \tilde{\psi} \circ \tilde{\phi}) $$
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**Theorem**

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Since homomorphisms are defined diagramatically, this is extremely simple to prove.
1. Dependent type theory

2. Natural models of type theory

3. Algebraic description of homomorphisms

4. Functorial description of homomorphisms

5. Interpreting the syntax
Remark on Kan extension

Any functor $F : \mathcal{C} \to \mathcal{D}$ between small categories induces an adjunction $F_! \dashv F^*$ between presheaf categories

$$
\begin{array}{ccc}
\text{Set}^{\mathcal{D}^{\text{op}}} & \xleftarrow{\sim} & \text{Set}^{\mathcal{C}^{\text{op}}} \\
\downarrow F_! & & \downarrow F^* \\
\text{Set}^{\mathcal{D}^{\text{op}}} & \xrightarrow{\sim} & \text{Set}^{\mathcal{C}^{\text{op}}}
\end{array}
$$

where
- $F^* = - \circ F$ is precomposition with $F$; and
- $F_!$ is left Kan extension along $F$. 

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Categories of natural models of type theory — CT 2016
Remark on Kan extension

Any functor $F : \mathcal{C} \to \mathcal{D}$ between small categories induces an adjunction $F_! \dashv F^*$ between presheaf categories

\[
\begin{array}{ccc}
\text{Set}^{\mathcal{C}^{\text{op}}} & \xrightarrow{F_!} & \text{Set}^{\mathcal{D}^{\text{op}}} \\
\uparrow{y} & \quad & \quad \uparrow{y} \\
\mathcal{C} & \xrightarrow{F} & \mathcal{D}
\end{array}
\]

where

- $F^* = - \circ F$ is precomposition with $F$; and
- $F_!$ is left Kan extension along $F$.

Moreover, $F_! \circ y \cong y \circ F : \mathcal{C} \to \text{Set}^{\mathcal{D}^{\text{op}}}$. 
Functorial presentation of homomorphisms

Specifying a homomorphism \((F, \Phi, \tilde{\Phi}) : \mathcal{C} \rightarrow \mathcal{D}\) is equivalent to specifying:

\[ \begin{align*}
F &: \mathcal{C} \rightarrow \mathcal{D} \\
\text{Natural transformations } \phi &: F_{\mathcal{U}} \rightarrow V \\
\tilde{\phi} &: F_{\tilde{\mathcal{U}}} \rightarrow \tilde{V} \\
\text{such that } F(\cdot) &= \cdot \\
\text{The diagram } F_{\mathcal{U}} \tilde{\mathcal{V}} &= \begin{array}{c}
\phi \\
\tilde{\phi}
\end{array} \\
F_{\mathcal{U}} V &= \begin{array}{c}
\phi \\
\tilde{\phi}
\end{array} \\
\text{for all } \Gamma &\in \text{ob}(\mathcal{C}) \\
\text{Comparison morphisms } c_{\Gamma A} &: F(\Gamma \cdot A) \rightarrow F_{\Gamma} \cdot FA &= \text{identities.}
\end{align*} \]
Functorial presentation of homomorphisms

Specifying a homomorphism \((F, \Phi, \tilde{\Phi}) : \mathcal{C} \to \mathcal{D}\) is equivalent to specifying:

- A functor \(F : \mathcal{C} \to \mathcal{D}\);
Functorial presentation of homomorphisms

Specifying a homomorphism $(F, \Phi, \tilde{\Phi}) : \mathcal{C} \to \mathcal{D}$ is equivalent to specifying:

- A functor $F : \mathcal{C} \to \mathcal{D}$;
- Natural transformations $\varphi : F_iU \to \mathcal{V}$ and $\tilde{\varphi} : F_i\tilde{U} \to \tilde{\mathcal{V}}$
Functorial presentation of homomorphisms

Specifying a homomorphism \((F, \Phi, \tilde{\Phi}) : \mathcal{C} \to \mathcal{D}\) is equivalent to specifying:

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such that

- \(F(\diamond) = \bullet\).
Functorial presentation of homomorphisms

Specifying a homomorphism \((F, \Phi, \tilde{\Phi}) : \mathcal{C} \to \mathcal{D}\) is equivalent to specifying:

- A functor \(F : \mathcal{C} \to \mathcal{D}\);
- Natural transformations \(\varphi : F_U \to V\) and \(\tilde{\varphi} : F_{\tilde{U}} \to \tilde{V}\) such that
- \(F(\Diamond) = \bullet\);
- The diagram
  \[
  \begin{array}{ccc}
  F_{\tilde{U}} & \xrightarrow{\tilde{\varphi}} & \tilde{V} \\
  F_{U} & \xrightarrow{\varphi} & V
  \end{array}
  \]
  commutes;
- The comparison morphisms \(c_{\Gamma A} : F(\Gamma \cdot A) \to F_{\Gamma} \cdot FA\) are identities.
Functorial presentation of homomorphisms

Specifying a homomorphism \((F, \Phi, \tilde{\Phi}) : \mathcal{C} \to \mathcal{D}\) is equivalent to specifying:

- A functor \(F : \mathcal{C} \to \mathcal{D}\);
- Natural transformations \(\varphi : F_! \mathcal{U} \to \mathcal{V}\) and \(\tilde{\varphi} : F_! \tilde{\mathcal{U}} \to \tilde{\mathcal{V}}\) such that
  - \(F(\diamond) = \bullet\);
  - The diagram \(\begin{array}{ccc} F_! \tilde{\mathcal{U}} & \xrightarrow{\tilde{\varphi}} & \tilde{\mathcal{V}} \\ F_! p \downarrow & & \downarrow q \\ F_! \mathcal{U} & \xrightarrow{\varphi} & \mathcal{V} \end{array}\) commutes;
  - \(F(\Gamma \cdot A) = F\Gamma \cdot FA\) for all \(\Gamma \in \text{ob}(\mathcal{C})\); and
Functorial presentation of homomorphisms

Specifying a homomorphism \((F, \Phi, \widetilde{\Phi}) : \mathcal{C} \rightarrow \mathcal{D}\) is equivalent to specifying:

- A functor \(F : \mathcal{C} \rightarrow \mathcal{D}\);
- Natural transformations \(\varphi : F_U \rightarrow \mathcal{V}\) and \(\tilde{\varphi} : F_{\widetilde{U}} \rightarrow \tilde{\mathcal{V}}\) such that
  - \(F(\Diamond) = \bullet\);
  - The diagram \(\begin{array}{ccc}
  F_{\tilde{U}} & \xrightarrow{\tilde{\varphi}} & \tilde{\mathcal{V}} \\
  \downarrow F_{p} & & \downarrow q \\
  F_{U} & \xrightarrow{\varphi} & \mathcal{V}
  \end{array}\) commutes;
- \(F(\Gamma \cdot A) = F\Gamma \cdot FA\) for all \(\Gamma \in \text{ob}(\mathcal{C})\); and
- The comparison morphisms \(c^\Gamma_A : F(\Gamma \cdot A) \rightarrow F\Gamma \cdot FA\) are identities.
Action on types and terms

We obtain an action of $\varphi$ on types and $\tilde{\varphi}$ on terms as follows.
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- **Action on types.** $A \in \mathcal{U}(\Gamma) \leadsto FA \in \mathcal{V}(F\Gamma)$ via
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- **Action on types.** $A \in \mathcal{U}(\Gamma) \leadsto FA \in \mathcal{V}(F\Gamma)$ via

  \[
  y(\Gamma) \xrightarrow{A} \mathcal{U}
  \]
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We obtain an action of $\varphi$ on types and $\tilde{\varphi}$ on terms as follows.

- **Action on types.** $A \in U(\Gamma) \rightsquigarrow FA \in V(F\Gamma)$ via

  $$y(\Gamma) \xrightarrow{A} U \quad F_! y(\Gamma) \xrightarrow{F_! A} F_! U$$
Action on types and terms

We obtain an action of \( \varphi \) on types and \( \tilde{\varphi} \) on terms as follows.

\begin{itemize}
  \item **Action on types.** \( A \in \mathcal{U}(\Gamma) \leadsto FA \in \mathcal{V}(F\Gamma) \) via
    \[
    \begin{align*}
    y(\Gamma) & \xrightarrow{A} \mathcal{U} \\
    F_{!}y(\Gamma) & \xrightarrow{F_{!}A} F_{!}\mathcal{U} \quad \varphi \xrightarrow{=} \mathcal{V}
    \end{align*}
    \]
\end{itemize}
Action on types and terms

We obtain an action of $\varphi$ on types and $\tilde{\varphi}$ on terms as follows.

- **Action on types.** $A \in \mathcal{U}(\Gamma) \rightsquigarrow FA \in \mathcal{V}(F\Gamma)$ via

$$
\begin{align*}
\begin{array}{ccc}
y(\Gamma) & \xrightarrow{A} & \mathcal{U} \\
F_!y(\Gamma) & \xrightarrow{F_!A} & F_!\mathcal{U} & \xrightarrow{\varphi} & \mathcal{V}
\end{array}
\end{align*}
$$
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- **Action on types.** $A \in \mathcal{U}(\Gamma) \leadsto FA \in \mathcal{V}(F\Gamma)$ via

\[
\begin{align*}
\text{y}(\Gamma) & \xrightarrow{A} \mathcal{U} \\
\mathcal{U} & \xrightarrow{\varphi} \mathcal{V}
\end{align*}
\]

\[
\begin{align*}
F\text{y}(\Gamma) & \xrightarrow{F_{!}A} F_{!}\mathcal{U} \\
F_{!}\mathcal{U} & \xrightarrow{\varphi} \mathcal{V}
\end{align*}
\]
Action on types and terms

We obtain an action of $\varphi$ on types and $\tilde{\varphi}$ on terms as follows.

- **Action on types.** $A \in U(\Gamma) \sim \Rightarrow FA \in V(F\Gamma)$ via

\[
\begin{align*}
  y(\Gamma) & \xrightarrow{A} U \\
  F\varphi y(\Gamma) & \xrightarrow{FA} F\varphi U \xrightarrow{\varphi} V \\
  \models & \xrightarrow{\sim} \\
  y(F\Gamma) & \xrightarrow{FA}
\end{align*}
\]

- **Action on terms.** $a \in \tilde{U}(\Gamma) \sim \Rightarrow Fa \in \tilde{V}(F\Gamma)$ via

\[
\begin{align*}
  y(\Gamma) & \xrightarrow{a} \tilde{U} \\
  F\tilde{\varphi} y(\Gamma) & \xrightarrow{Fa} F\tilde{\varphi} \tilde{U} \xrightarrow{\tilde{\varphi}} \tilde{V} \\
  \models & \xrightarrow{\sim} \\
  y(F\Gamma) & \xrightarrow{Fa}
\end{align*}
\]
Where the comparison morphisms $c_A^\Gamma$ come from

Set-up: $A$ type in context $\Gamma$
Where the comparison morphisms $c^\Gamma_A$ come from

\[
y(\Gamma \cdot A) \xrightarrow{q^\Gamma_A} \tilde{U} \\
y(p^\Gamma_A) \downarrow \quad \downarrow p \\
y(\Gamma) \xrightarrow{A} U
\]

Context extension of $\Gamma$ by $A$
Where the comparison morphisms $c^\Gamma_A$ come from

Apply $F_!$
Where the comparison morphisms $c^\Gamma_A$ come from

\[
\begin{align*}
  y(F(\Gamma \cdot A)) & \xrightarrow{\mathcal{R}} F_!y(\Gamma \cdot A) & \xrightarrow{F_!q_A^\Gamma} & F_!\tilde{U} \\
  y(Fp_A) & \downarrow & F_!y(p_A^\Gamma) & \downarrow & F_!(p) \\
  y(F\Gamma) & \xrightarrow{\mathcal{R}} F_!y(\Gamma) & \xrightarrow{F_!A} & F_!U
\end{align*}
\]

\[F_! \circ y \simeq y(F-)\]
Where the comparison morphisms $c^A_{\Gamma}$ come from

\[ y(F(\Gamma \cdot A)) \xrightarrow{\sim} F_1 y(\Gamma \cdot A) \xrightarrow{F_1 q^A_{\Gamma}} F_1 \tilde{U} \xrightarrow{\varphi} \tilde{V} \]

\[ y(Fp^A_{\Gamma}) \downarrow \quad F_1 y(p^A_{\Gamma}) \downarrow \quad F_1 y(\Gamma) \downarrow F_1 A \]

\[ y(F\Gamma) \xrightarrow{\sim} F_1 y(\Gamma) \xrightarrow{F_1 A} F_1 U \xrightarrow{\varphi} V \]

Paste square for $\varphi$, $\tilde{\varphi}$
Where the comparison morphisms $c_A^\Gamma$ come from

Action of $\varphi$ on types and $\tilde{\varphi}$ on terms
Where the comparison morphisms $c^\Gamma_A$ come from

\[ y(F\Gamma \cdot FA) \]
\[ y(F(\Gamma \cdot A)) \]
\[ y(F\Gamma) \]

Extend context $F\Gamma$ by $FA$
Where the comparison morphisms $c_A^\Gamma$ come from

\[ y(F\Gamma \cdot FA) \]
\[ \Downarrow \]
\[ y(F\Gamma) \]
\[ \Downarrow \]
\[ y(Fp_A^\Gamma) \]
\[ \Downarrow \]
\[ y(\Gamma \cdot A) \]
\[ \Downarrow \]
\[ y(F\Gamma) \]
\[ \Downarrow \]
\[ y(F\Gamma \cdot FA) \]
\[ \Downarrow \]
\[ \sim \]
\[ Fq_A^\Gamma \]
\[ q_{FA} \]
\[ \sim \]
\[ FA \]

Obtain $c_A^\Gamma : F(\Gamma \cdot A) \to F\Gamma \cdot FA$ as shown
Where the comparison morphisms $c^\Gamma_A$ come from

\[
y(F\Gamma \cdot FA) \quad y(F(\Gamma \cdot A)) \quad y(F\Gamma) \quad y(F\Gamma)
\]

\[
q_{FA}^\Gamma \\
Fq_{A}^\Gamma \\
FA \\
\sim = \quad \sim =
\]

\[
c^\Gamma_A = \text{id} \quad \Rightarrow \quad Fp_{A}^\Gamma = p_{FA}^F \quad \text{and} \quad Fq_{A}^\Gamma = q_{FA}^F.
\]
1. Dependent type theory
2. Natural models of type theory
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Interpreting the syntax

We take a similar approach to that of S. Castellan, P. Clairambault, P. Dybjer (2015). The idea is as follows:
Interpreting the syntax

We take a similar approach to that of S. Castellan, P. Clairambault, P. Dybjer (2015). The idea is as follows:

- Work in a system of type theory with four kinds of judgements

\[ \Gamma = \Gamma' \vdash, \quad \Delta \vdash \gamma = \gamma' : \Gamma, \quad \Gamma \vdash A = A', \quad \Gamma \vdash a = a' : A \]

(We write \( \Gamma \vdash \) instead of \( \Gamma = \Gamma \vdash \), and so on.)
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- From the syntax, build a natural model

\[
\mathcal{T} = (\mathbb{T}, [], \text{Ty}, \text{Tm}, \text{ty}, \text{ext}, \text{sub}, \text{proj})
\]

called the \textit{term model} of the system.
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- From the syntax, build a natural model

\[ \mathcal{T} = (\Gamma, [], \text{Ty}, \text{Tm}, \text{ty}, \text{ext}, \text{sub}, \text{proj}) \]

called the term model of the system.

- \( \mathcal{T} \) will (in a suitable sense) be the free natural model supporting the derivation rules for this system.
Example 1: basic syntax

With no rules for type formation, the term model is very simple:
Example 1: basic syntax

With no rules for type formation, the term model is very simple:

- $\mathbb{T}$ has the empty context $[]$ as its only object and the identity substitution $[] \vdash \text{id} : []$ as its only morphism;

- $\mathsf{Ty}, \mathsf{Tm} : \mathbb{T}^{\text{op}} \to \mathbf{Set}$ are the empty presheaves;
- $\mathsf{ty}, \mathsf{ext}$ are the unique (empty) functor between empty categories;
- $\mathsf{sub}, \mathsf{proj}$ are the unique natural transformations with no components.

It is very easy to prove the following result.

**Theorem**

This data defines a natural model $\mathbb{T}$, which is an initial object in $\mathsf{NM}$. 

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It is very easy to prove the following result.

**Theorem**

*This data defines a natural model $\mathcal{T}$, which is an initial object in $\textbf{NM}$.*
Example 2: adding a unit type

Consider the type theory obtained by adding a unit type 1, i.e. we add the following rules to our syntax:

\[
\begin{align*}
\vdash 1 \\
\vdash \ast : 1 \\
\vdash a : 1 \\
\vdash a = \ast : 1
\end{align*}
\]

The term model \( \mathcal{T} \) for this system is defined as follows:

...
Example 2: adding a unit type

Consider the type theory obtained by adding a unit type \( 1 \), i.e. we add the following rules to our syntax:

\[
\begin{align*}
\Gamma & \vdash 1 \\
\Gamma & \vdash \star : 1 \\
\Gamma & \vdash a : 1 \\
\Gamma & \vdash a = \star : 1
\end{align*}
\]

The term model \( \mathcal{T} \) for this system is defined as follows:

- The objects of \( \mathcal{T} \) are the empty context \([0] := []\) and finite strings of the form \([n] := [\underbrace{1 \cdot 1 \cdots 1}_n]\) for \( n \geq 1 \);

\( n \) times
Example 2: adding a unit type

Consider the type theory obtained by adding a unit type $1$, i.e. we add the following rules to our syntax:

$$
\begin{align*}
\vdash 1 \\
\vdash \star : 1 \\
\vdash a : 1 \\
\vdash a = \star : 1
\end{align*}
$$

The term model $\Sigma$ for this system is defined as follows:

- The objects of $\mathbb{T}$ are the empty context $[0] := []$ and finite strings of the form $[n] := [1 \cdot 1 \cdots 1]$ for $n \geq 1$; $n$ times.
- There is a unique morphism $\gamma_{n,m} : [n] \to [m]$ for all $n, m \in \mathbb{N}$. 

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Categories of natural models of type theory — CT 2016
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\Gamma & \vdash a = \ast : 1
\end{align*}
\]

The term model \( \mathcal{S} \) for this system is defined as follows:

- The objects of \( \mathbb{T} \) are the empty context \([0] := []\) and finite strings of the form \([n] := [1 \cdot 1 \cdots 1]\) for \( n \geq 1 \); 
  \( n \) times
- There is a unique morphism \( \gamma_{n,m} : [n] \rightarrow [m] \) for all \( n, m \in \mathbb{N} \).
- \( \text{Ty}([n]) = \{[1]\} \) and \( \text{Ty}(\gamma_{n,m}) = \text{id}_{\{[1]\}} \).
Example 2: adding a unit type

Consider the type theory obtained by adding a unit type \( \mathbf{1} \), i.e. we add the following rules to our syntax:

\[
\begin{align*}
\vdash \mathbf{1} \\
\vdash \ast : \mathbf{1} \\
\vdash a : \mathbf{1} \\
\vdash a = \ast : \mathbf{1}
\end{align*}
\]

The term model \( \mathcal{T} \) for this system is defined as follows:

- The objects of \( \mathcal{T} \) are the empty context \([0] := [\ ]\) and finite strings of the form \([n] := [1 \cdot 1 \cdots 1]\) for \( n \geq 1; \)
  \( n \) times
- There is a unique morphism \( \gamma_{n,m} : [n] \to [m] \) for all \( n, m \in \mathbb{N} \).
- \( \text{Ty}([n]) = \{[1]\} \) and \( \text{Ty}(\gamma_{n,m}) = \text{id}_{\{[1]\}} \);
- \( \text{Tm}([n]) = \{[\ast]\} \) and \( \text{Tm}(\gamma_{n,m}) = \text{id}_{\{[\ast]\}} \);
Example 2: adding a unit type

Consider the type theory obtained by adding a unit type 1, i.e. we add the following rules to our syntax:

\[ \vdash 1 \]
\[ \vdash \star : 1 \]
\[ \vdash a : 1 \]
\[ a = \star : 1 \]

The term model \( \mathcal{T} \) for this system is defined as follows:

- The objects of \( \mathcal{T} \) are the empty context \([0] := []\) and finite strings of the form \([n] := [1 \cdot 1 \cdots 1] \) for \( n \geq 1 \);
  
- There is a unique morphism \( \gamma_{n,m} : [n] \rightarrow [m] \) for all \( n, m \in \mathbb{N} \).
- \( Ty([n]) = \{[1]\} \) and \( Ty(\gamma_{n,m}) = \text{id}_{\{[1]\}} \);
- \( Tm([n]) = \{[\star]\} \) and \( Tm(\gamma_{n,m}) = \text{id}_{\{[\star]\}} \);
- \( ty([n], [\star]) = ([n], [1]) \) and \( \text{ext}([n], [1]) = ([n + 1], [\star]) \);
Example 2: adding a unit type

Consider the type theory obtained by adding a unit type $1$, i.e. we add the following rules to our syntax:

$$\vdash 1$$

$$\vdash \star : 1$$

$$\vdash a : 1$$

The term model $\mathcal{T}$ for this system is defined as follows:

- The objects of $\mathcal{T}$ are the empty context $[0] := []$ and finite strings of the form $[n] := [1 \cdot 1 \cdots 1]$ for $n \geq 1$; $n$ times
- There is a unique morphism $\gamma_{n,m} : [n] \rightarrow [m]$ for all $n, m \in \mathbb{N}$.
- $\text{Ty}([n]) = \{[1]\}$ and $\text{Ty}(\gamma_{n,m}) = \text{id}_{\{[1]\}}$;
- $\text{Tm}([n]) = \{[\star]\}$ and $\text{Tm}(\gamma_{n,m}) = \text{id}_{\{[\star]\}}$;
- $\text{ty}([n], [\star]) = ([n], [1])$ and $\text{ext}([n], [1]) = ([n + 1], [\star])$;
- $\text{sub}([n], [\star]) = \gamma_{n,n+1}$ and $\text{proj}([n], [\star]) = \gamma_{n+1,n}$.
Example 2: adding a unit type

A natural model $\mathcal{C}$ supports the unit type if there exist

$$1_{\mathcal{C}} \in \mathcal{U}(\diamond) \quad \text{and} \quad \star_{\mathcal{C}} \in \mathcal{\tilde{U}}(\diamond)$$

such that $p_\diamond^{-1}(\{1_{\mathcal{C}}\}) = \{\star_{\mathcal{C}}\}$.
Example 2: adding a unit type

A natural model \( \mathcal{C} \) supports the unit type if there exist

\[ 1_\mathcal{C} \in \mathcal{U}(\Diamond) \quad \text{and} \quad \ast_\mathcal{C} \in \tilde{\mathcal{U}}(\Diamond) \]

such that \( p_\Diamond^{-1}(\{1_\mathcal{C}\}) = \{\ast_\mathcal{C}\} \).

Theorem

The data \( \mathcal{Z} \) on the previous slide defines a natural model, which is the free natural model supporting the unit type, i.e.
Example 2: adding a unit type

A natural model \( \mathfrak{c} \) supports the unit type if there exist

\[
1_{\mathfrak{c}} \in \mathcal{U}(\diamond) \quad \text{and} \quad \star_{\mathfrak{c}} \in \tilde{\mathcal{U}}(\diamond)
\]

such that \( p^{-1}_\diamond(\{1_{\mathfrak{c}}\}) = \{\star_{\mathfrak{c}}\} \).

Theorem

The data \( \mathfrak{T} \) on the previous slide defines a natural model, which is the free natural model supporting the unit type, i.e. if \( \mathfrak{c} \) is any natural model supporting the unit type, and \( 1_{\mathfrak{c}} \in \mathcal{U}(\diamond) \) and \( \star_{\mathfrak{c}} \in \tilde{\mathcal{U}}(\diamond) \) are as above, then there is a unique homomorphism

\[
(F, \Phi, \tilde{\Phi}) : \mathfrak{T} \rightarrow \mathfrak{c}
\]

such that

\[
F[1] = 1_{\mathfrak{c}} \quad \text{and} \quad F[\star] = \star_{\mathfrak{c}}
\]
Future work on natural models

In the pipeline:
Future work on natural models

In the pipeline:

- Uniform construction of the term model for a signature;
Future work on natural models

In the pipeline:
- Uniform construction of the term model for a signature;
- Lawvere duality for natural models;
Future work on natural models

In the pipeline:

■ Uniform construction of the term model for a signature;
■ Lawvere duality for natural models;
■ Investigation of the polynomial functor induced by \( p : \tilde{\mathcal{U}} \to \mathcal{U} \).
## Acknowledgements and references

This is joint work with my PhD advisor, Steve Awodey.

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Introduction and basic theory of natural models:

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Introduction and basic theory of natural models:


Some analogous results for categories with families:

- S. Castellan, *Categories with families as the initial category with families* (2014).
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Thank you for listening!