Natural models and tricategories of polynomials
HoTT MURI meeting

Clive Newstead
Carnegie Mellon University

Friday 24th March 2017
1. Polynomials in locally cartesian closed categories
2. Natural models of type theory
3. Tricategories of polynomials
4. Type theory is a pseudomonad
1. Polynomials in locally cartesian closed categories

2. Natural models of type theory

3. Tricategories of polynomials

4. Type theory is a pseudomonad
A **locally cartesian closed category** is a category $\mathcal{E}$ with a terminal object and a chain of adjoint functors $\Sigma_f \dashv f^* \dashv \Pi_f$ for each $f : B \to A$ in $\mathcal{E}$:

\[
\begin{tikzcd}
\mathcal{E}/A & \mathcal{E}/B \\
\downarrow & \\
\downarrow & \\
(\Sigma_f \downarrow f^* \downarrow \Pi_f)
\end{tikzcd}
\]

where $\Sigma_f = f \circ (\_)$ and $f^*$ is pullback along $f$. 
Internal language of a LCCC

- Objects $x : X \rightarrow A$ of $\mathcal{E}/A \leadsto (X_a \mid a \in A)$
- Morphisms $f : x \rightarrow y$ in $\mathcal{E}/A \leadsto (f_a : X_a \rightarrow Y_a \mid a \in A)$. 

$$\Sigma_f(X_b \mid b \in B) = \left( \sum_{b \in B_a} X_{f(a)} \mid a \in A \right)$$

$$\Pi_f(X_b \mid b \in B) = \left( \prod_{b \in B_a} X_{f(a)} \mid a \in A \right)$$

$$f^*(Y_a \mid a \in A) = \left( Y_{f(b)} \mid b \in B \right)$$
A polynomial $F$ in a LCCC $\mathcal{E}$ is a diagram of the form:

$$
\begin{array}{c}
B & \xrightarrow{f} & A \\
\downarrow{s} & & \downarrow{t} \\
I & & J \\
\end{array}
$$

We take $I = J = 1 \rightsquigarrow$ identify with morphisms $f : B \to A$.

Think: $f : B \to A = $ signature for an algebraic theory: $B_a$-ary operation for each $a \in A$. 
Polynomial functors

Let $f : B \to A$ be a polynomial. The **extension** of $f$ is the functor

$$P_f = \Sigma_A \circ \Pi_f \circ B^* : \mathcal{E} \to \mathcal{E}$$

In the internal language:

$$P_f(X) = \sum_{a \in A} \prod_{b \in B_a} X = \sum_{a \in A} X^{B_a}$$

Polynomial functors are closed under composition.
Morphisms of polynomials

A morphism of polynomials from $f : B \to A$ to $g : D \to C$ is $(\varphi, \tilde{\varphi})$ as in:

![Diagram](image.png)

Morphisms of polynomials $\varphi : f \Rightarrow g$ induce cartesian natural transformations $P_\varphi : P_f \Rightarrow P_g$ between polynomial functors.

Clive Newstead (cnewstead@cmu.edu)
Carnegie Mellon University

Natural models and tricategories of polynomials
A bicategory of polynomials

There is a bicategory $\text{Poly}_E$ whose 0-cells are the objects of $E$, whose 1-cells are polynomials and whose 2-cells are morphisms of polynomials;

There is a 2-category $\text{PolyFun}_E$ whose 0-cells are the slices of $E$, whose 1-cells are polynomial functors and whose 2-cells are cartesian natural transformations.

The assignments $I \mapsto E/I$, $F \mapsto P_F$ and $\varphi \mapsto P_\varphi$ induce a biequivalence of bicategories.
<table>
<thead>
<tr>
<th>1</th>
<th>Polynomials in locally cartesian closed categories</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Natural models of type theory</td>
</tr>
<tr>
<td>3</td>
<td>Tricategories of polynomials</td>
</tr>
<tr>
<td>4</td>
<td>Type theory is a pseudomonad</td>
</tr>
</tbody>
</table>
Natural models of type theory

A natural model is a representable natural transformation

\[ p : \tilde{\mathcal{U}} \rightarrow \mathcal{U} \]

between presheaves on a category \( \mathcal{C} \).

Take \( \mathcal{C} = \) syntactic category of DTT \( \mathbb{T} \). Then:

- \( \mathcal{U} = \) presheaf of types
- \( \tilde{\mathcal{U}} = \) presheaf of terms
- \( p \) assigns to each term its type
Supporting unit types

A natural model $p$ supports a unit type $\iff$ there exist $\mathbb{1}, \ast$ such that:

$$
\begin{align*}
  & y(\diamond) \quad \ast \quad \tilde{U} \\
  & \Downarrow \quad \Downarrow \quad \quad \quad \Downarrow \quad \quad \quad \Downarrow \quad \quad \quad \Downarrow \\
  & y(\diamond) \quad \mathbb{1} \quad U \\
  \end{align*}
$$

$\leadsto \eta : \mathbb{1} \Rightarrow p$
Supporting dependent sum types

A natural model $p$ supports dependent sum types $\iff$ there exist $\tilde{\Sigma}$, pair making the following square a pullback:

\[
\begin{array}{ccc}
\sum_{(A,B)} \sum_{a:A} B(A) & \xrightarrow{\text{pair}} & \tilde{U} \\
\downarrow & & \downarrow \\
\sum_{A:U} U^A & \xrightarrow{\tilde{\Sigma}} & U \\
\end{array}
\]

$\sim \mu : p \cdot p \Rightarrow p$
Is it a monad?

**Conclusion:** $p$ supports a unit type and dependent sum types $\iff$ there exist morphisms of polynomials

$$\eta : 1 \Rightarrow p \quad \text{and} \quad \mu : p \cdot p \Rightarrow p$$

**Problem:** not quite monad—unit and associativity not strict.

**Solution:** Add a new dimension of structure.
1. Polynomials in locally cartesian closed categories
2. Natural models of type theory
3. Tricategories of polynomials
4. Type theory is a pseudomonad
2-bicategories

A **2-bicategory** \( \mathcal{B} \) has 0-, 1-, 2- and 3-cells, such that...

- The 0-, 1- and 2- cells form a bicategory;
- For fixed \( I, J \), the 1-cells from \( I \) to \( J \), 2-cells and 3-cells form a 2-category;
- Composition and coherence isomorphisms behave nicely.

All 2-bicategories are tricategories, and all 3-categories are 2-bicategories.

**Goal:** Equip bicategory \( \text{Poly}_E \) with 3-cells turning it into a 2-bicategory.
Internal full subcategories

There is a functor $\text{Poly}_\mathcal{E}(1, 1) \to \text{Cat}(\mathcal{E})$:

- $f : B \to A \rightsquigarrow$ internal category $\mathbb{A}_f$, with
  
  $\mathbb{A}_{f,0} = A$,  \quad $\text{Hom}_{\mathbb{A}_f} = (B_{a a'}^B | a, a' \in A)$

- $\varphi : f \Rightarrow g \rightsquigarrow$ internal functor

  $F_{\varphi} : \mathbb{A}_f \to \mathbb{A}_g$

**Idea:** 3-cells should be particular internal natural transformations.
Adjustments

An adjustment $\alpha : \varphi \Rightarrow \psi : f \Rightarrow g$ is $\alpha : \varphi^* g \to \psi^* g$ in $E/A$ as in:

\[ \begin{array}{ccc}
D\varphi & \xrightarrow{\alpha} & D\psi \\
\downarrow & & \downarrow \\
\varphi^* g & \sim & \psi^* g \\
\varphi & \searrow & \psi \\
\varphi^* g & \downarrow & \psi^* g \\
f & \downarrow & f \\
A & \leftarrow & A
\end{array} \]

\[ \bar{\varphi} \sim \bar{\psi} \]

\[ \bar{\varphi} \sim \bar{\psi} \sim f \]

... where $\bar{\varphi}$ and $\bar{\psi}$ are the canonical isomorphisms arising from the universal property of pullbacks.
Adjustments

In the internal language of $\mathcal{E}$, an adjustment $\alpha : \varphi \Rightarrow \psi$ is

$$\left( \alpha_a : D\varphi(a) \rightarrow D\psi(a) \mid a \in A \right)$$

+ naturality

Adjustments do indeed induce internal natural transformations.
Almost there...

**Theorem**

There is a 2-bicategory $\mathcal{P}ol_\mathcal{E}$ with

- **0-cells** = objects of $\mathcal{E}$
- **1-cells** = polynomials
- **2-cells** = morphisms of polynomials
- **3-cells** = adjustments

Moreover, the 1-functor $\text{Poly}_\mathcal{E}(1, 1) \to \text{Cat}(\mathcal{E})$ extends to a 2-functor $\mathcal{P}ol_\mathcal{E}(1, 1) \to \text{Cat}(\mathcal{E})$. 
1. Polynomials in locally cartesian closed categories
2. Natural models of type theory
3. Tricategories of polynomials
4. Type theory is a pseudomonad
Main result

Theorem
Let $p : \tilde{U} \to U$ be a natural model. Then $p$ supports a unit type and dependent sum types if and only if $p$ is the underlying 1-cell of a pseudomonad in the 2-bicategory $\text{Poly}_E$.

Proof.
We know $p$ supports unit and dependent sum types iff there exist $\eta : 1 \Rightarrow P$ and $\mu : p \cdot p \Rightarrow p$. Remains to show the monad laws hold up to invertible 3-cells.

e.g. Define $\lambda : \mu \circ (p \cdot \eta) \Rightarrow \text{id}_p$:

$$(\lambda_A : \tilde{U}_{1 \times A} \to \tilde{U}_A \mid A \in U)$$

Take $\lambda_A(\langle \star, a \rangle) = a$.

\ldots and so on.
Type theory is an algebra for itself(?)

A natural model $p$ supports dependent product types $\iff$ there exist $\hat{\Pi}, \hat{\lambda}$ making the following square a pullback:

$$
\begin{array}{ccc}
\sum_{A:U} \tilde{U}^A & \xleftarrow{\lambda} & \tilde{U} \\
\downarrow & & \downarrow p \\
\sum_{A:U} U^A & \xrightarrow{\Pi} & U \\
\end{array}
$$

This looks like an algebra for the pseudomonad. Work in progress.
References


