An infinite descent into pure mathematics

ACOG seminar, University of Pittsburgh

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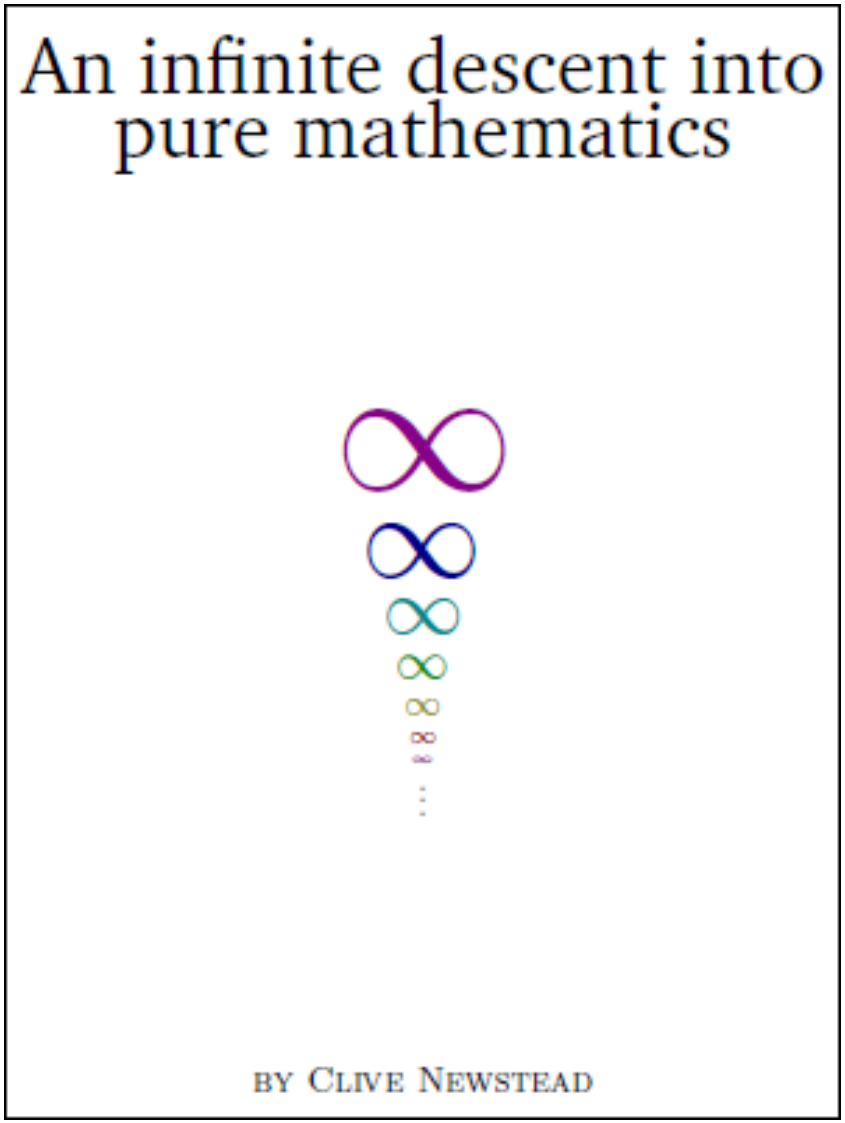

An infinite descent into pure mathematics
BY CLIVE NEWSTEAD

Traditional mathematics class

Course design

- Choose topics to be covered
- Choose grade weightings for assessments
- Choose textbook & set schedule

Classes

- Lecturer presents content
- Maybe some question-and-answer interaction

Assessment

- Problem sheets
- Quizzes and examinations

Current research about teaching and learning

Good practices to maximise student learning:

- Proper **alignment** of learning objectives, teaching strategies and methods of assessment
- Engaging students **actively** in the learning process
- Use of **inquiry-based** strategies and assessments
- Using a **variety** of teaching methods and assessments

Reference: *How Learning Works* (2010) by Susan Ambrose & co.

Alternative teaching model

Course design

- Choose learning objectives
- Design classes around helping students achieve LOs
- Choose types of assessment that test these LOs

Classes

- Before: some content delivery (e.g. reading, video)
- During: activities targetting LOs, clarifications, some lecture

Assessment

- Problem sheets, quizzes, examinations
- Projects, presentations, group work, . . .

Concepts of Mathematics — description

From course catalogue:

> *"This course introduces the basic concepts, ideas and tools involved in **doing mathematics**. As such, its main focus is on **presenting informal logic**, and the **methods of mathematical proof**. [...list of topics...]"*

From department website:

> *"Truth values, connectives, truth tables, contrapositives. Quantifiers. Proof by contradiction. Sets, intersections, unions, differences, the empty set. Integers, divisibility. Proof by induction. Primes, sieve of Eratosthenes, prime factorization. Gcd and lcm, Euclid's algorithm, solving $ax + by = c$. Congruences, modular arithmetic. Recursion. Linear recurrences. Functions and inverses. Permutations. Binomial coefficients, Catalan number. Inclusion-exclusion. Infinite cardinalities. Binary operations. Groups. Binary relations, equivalence relations. Graphs. Euler characteristic, planar graphs, five color theorem, rationals, reals, polynomials, complex numbers."*

Concepts of Mathematics — course design

Learning objectives

- "Presenting informal logic" = **communication**
- "Methods of mathematical proof" = **problem-solving**

Mathematical topics

- Symbolic logic, sets, functions
- Induction on the natural numbers
- Number theory
- Combinatorics
- Other topics *(real numbers, probability theory, basic topology, ...)*

Syllabus: ✓ Next step: **find a textbook**.

Concepts of Mathematics — textbook

Textbook criteria: a textbook for Concepts should. . .

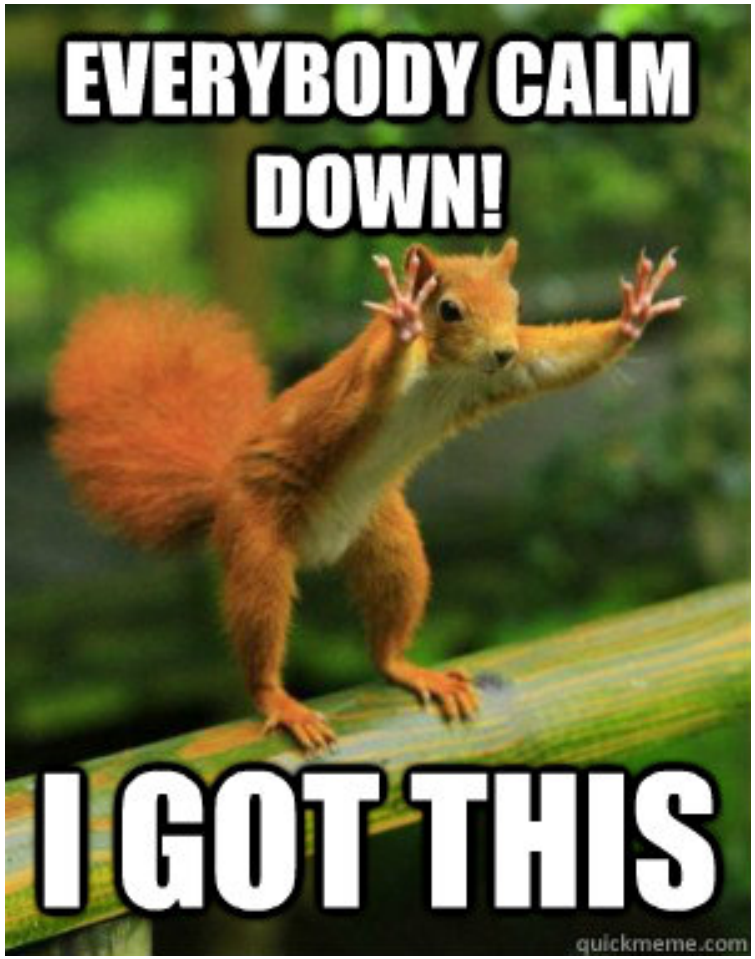
- Be of an appropriate length
- Cover enough mathematical topics
- Cover communication and problem-solving skills
- Practise what it preaches
- Be as agnostic as possible

Concepts of Mathematics — textbook

Solution: Write my own notes

Time frame: 51 days

Backup plan: Concede



Design considerations

Lots of decisions to make

- Mathematical areas to cover
- Definitions and theorems to emphasise
- Include exercise solutions or not?
- Level of difficulty
- Choices of convention
- Level of verbosity
- Level of detail in proofs
- General tone of the book
- How to cover both skills and content
- Name of the book
- Licensing and copyright issues

Example dilemma #1

What is a function?

Possible definitions:

(1) A set of ordered pairs such that . . .

(2) A triple (X, Y, f) where $f \subseteq X \times Y$ such that . . .

(3) A rule assigning to each x a unique y

(4) An imaginary machine taking inputs and giving outputs

(5) A primitive notion in terms of which all other mathematical notions are defined

My choice: (3) because it is the most agnostic

Example dilemma #2

Is zero a natural number?

Possible resolutions:

(1) Yes

(2) No

(3) Choose your own adventure, make explicit when needed

(4) Use $\mathbb{N}_0$ and $\mathbb{N}_1$ *(or similar)*

My choice: (1) for lots of reasons

Example dilemma #3

What is the best domain of discourse for number theory?

Possible answers:

(1) $\mathbb{Z}$

(2) $\mathbb{N}$ (with or without zero?)

(3) Sometimes $\mathbb{N}$, sometimes $\mathbb{Z}$

My choice: (1) because it generalises easily to more general rings

Return to learning objectives

Communication skills

- Using notation accurately
- Developing mathematical fluency
- Evaluating effectiveness of others' proofs
- Typesetting in LaTeX

Design principles

- Write accurately and clearly
- Include discussion exercises
- Provide guidance on how to structure a proof
- Provide LaTeX support

Return to learning objectives

Problem-solving skills

- Identifying feasible proof strategies
- Identifying relevant definitions and theorems
- Creativity in problem-solving approaches

Design principles

- Examples and exercises galore
- Vary level of difficulty
- Provide problem-solving tips
- Do not provide solutions to exercises

And so the writing began

Note: this is not actually me. . . I don't have a Mac

Growth of the book

Date	Event	# pages	Δ
May 9, 2015	Started writing	0	–
Jun 29, 2015	Started teaching 21-127	134	134
Aug 7, 2015	Finished teaching 21-127	183	49
May 24, 2016	Started preparing for 21-128	183	0
Aug 29, 2016	Started teaching 21-128	204	21
Dec 9, 2016	Finished teaching 21-128	348	144
Mar 1, 2018	*(today)*	394	46
?????	Finished writing	< 500	???

Next steps

Remaining tasks

- Finish remaining chapters
- Add more examples, discussions & exercises
- Add more guidance for communication and proof-writing
- Add more diagrams and graphics
- Include chapter introductions, reflections and summaries

Reflection

What I have learnt

- Writing a textbook takes a lot of time and effort
- Writing a textbook does not contribute towards PhD requirements
- You can't make everyone happy
- LaTeX is full of surprises
- Having a project to work on is fun
- This book might never be finished

Thanks for listening!

Website

www.infinitedescent.xyz

These slides

math.cmu.edu/~cnewstea/talks/20180301.pdf