# 15-151 Homework 10

Please submit in class at 8:00am on Wednesday 9th August

*If you want feedback on your completed project background section, then you must submit it in class at 8:00am on Tuesday 8th August*

**Exercises**

1. The goal of this question is to prove that the cartesian proudct of two countably infinite sets is countably infinite.

   (a) Prove that the function $e : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by

   $$e(m, n) = 2^m(2n + 1) - 1 \text{ for all } (m, n) \in \mathbb{N} \times \mathbb{N}$$

   is a bijection. **[4 points]**

   (b) Explain why part (a) implies that $\mathbb{N} \times \mathbb{N}$ is countably infinite. **[1 point]**

   (c) Suppose $X$ and $Y$ are countably infinite sets, and let $f : \mathbb{N} \to X$ and $g : \mathbb{N} \to Y$ be bijections. Define a new function $f \times g : \mathbb{N} \times \mathbb{N} \to X \times Y$ by

   $$(f \times g)(m, n) = (f(m), g(n)) \text{ for all } (m, n) \in \mathbb{N} \times \mathbb{N}$$

   Prove that $f \times g$ is a bijection. **[5 points]**

   (d) With sets and functions as defined in (a) and (c), find a bijection $\mathbb{N} \to X \times Y$ expressed in terms of the functions $e$, $f$ and $g$. **[3 points]**

   (e) Prove that $\mathbb{Z} \times \mathbb{Z}$ is countably infinite. **[2 points]**

2. Let $X$ be the set of all infinite sequences of 0s and 1s, i.e.

   $$X = \{(a_0, a_1, a_2, \dots) \mid a_k \in \{0, 1\} \text{ for all } k \in \mathbb{N}\}$$

   Prove that $X$ is uncountably infinite. **[10 points]**

*This assignment is continued on the next page.*

### Project milestones

Please submit your work for the project milestone separately from your homework.

- *(Optional but recommended.)* Write a list of the definitions and preliminary results that you intend to include in your project's background section. If you want to receive feedback on this, you should upload the list to the 'Course project background summary' assignment on Canvas by 6:00pm on Saturday 5th August.
- Write the background and main result sections of your project. **[10 points]**

*If you want feedback on your background and/or main result sections, you should hand this work in during class on Tuesday 8th August.*

### Optional problems

You do not need to attempt these problems. If you do, then your score out of 35 on these questions (plus the two optional problems from Homework 9) will replace your lowest homework score from Homeworks 1 through 8.

Opt 3. Let $n, k \in \mathbb{N}$. Find and prove an expression for the number of injective functions $[n] \to [k]$. **[7 points]**

Opt 4. Find a bijection $f : [0, 1) \to (0, 1)$, where

$$[0, 1) = \{x \in \mathbb{R} \mid 0 \leq x < 1\} \quad \text{and} \quad (0, 1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$$ **[7 points]**

Opt 5. Find a partition of $\mathbb{N}$ into countably many countably infinite sets. (That is, find a collection $X_0, X_1, X_2, X_3, \dots$ of subsets of $\mathbb{N}$ which are all countably infinite, are pairwise disjoint, and such that every natural number is an element of $X_n$ for some $n \in \mathbb{N}$.) **[7 points]**