15-151 Homework 2

Please submit in class at 8:00am on Tuesday 11th July

Exercises

1. Read the following poorly written, albeit mostly mathematically correct, proof that $\sqrt{2}$ is irrational.

> **Proposition.** The real number $\sqrt{2}$ is irrational.
>
> *Proof.* $\sqrt{2}$ is rational, $\sqrt{2} = \dfrac{a}{b}$, assume cancelled to lowest terms.
>
> $$2 = \frac{a^2}{b^2}$$
> $$2b^2 = a^2 \rightarrow a^2 \text{ is even} \rightarrow a \text{ is even}$$
> $$2b^2 = (2k)^2 = 4k^2$$
> $$b^2 = 2k^2 \rightarrow b^2 \text{ is even} \rightarrow b \text{ is even}$$
>
> a and b are even, so $\dfrac{a}{b}$ is not cancelled to lowest terms. Contradiction! So $\sqrt{2}$ is irrational. □

Identify at least five ways in which the proof that could be improved, and write your own version of the proof which improves on these aspects. **[10 points]**

2. Let r be a rational number, and let a and b be irrational numbers. Which of the following numbers is necessary irrational?

$$a + r \quad ar \quad a^r \quad r^a \quad a^b$$

Prove your claims. You may assume without proof that $\sqrt{2}$ is irrational, but if you want to assume any other real number is irrational, then you must prove it. **[10 points]**

More questions on the next page

The following questions require proof by (weak) induction, which will be covered in class on Monday. If you want to get ahead, the relevant material in the book can be found in Section 1.3, up to page 58.

3. Prove by induction that
$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
for all $n \in \mathbb{N}$. **[5 points]**

4. Prove that $(1+x)^{15151} \geq 1 + 15151x$ for all $x \in \mathbb{R}$ with $x \geq -1$. **[5 points]**

5. Find and prove a formula for the following product:
$$\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1+\frac{(-1)^{n+1}}{n}\right)$$
where n varies over natural numbers greater than or equal to 2. **[5 points]**