15-151 Homework 6

Please submit in class at 8:00am on Tuesday 25th July

Exercises

1. Let $S : \mathbb{Z} \to \mathcal{P}(\mathbb{Z})$ be any function, and define a subset $B \subseteq \mathbb{Z}$ by

   $$B = \{n \in \mathbb{Z} \mid n \notin S(n)\}$$

   Prove that there is no $k \in \mathbb{Z}$ such that $B = S(k)$.

   Hint: suppose $B = S(k)$ for some $k \in \mathbb{Z}$, and consider whether $k \in B$. **[6 points]**

2. Let A, B, X be sets such that $A \cap B = \varnothing$, and let $f : A \to X$ and $g : B \to X$ be functions. Define $h : A \cup B \to X$ by letting

   $$h(t) = \begin{cases} f(t) & \text{if } t \in A \\ g(t) & \text{if } t \in B \end{cases}$$

   Prove that $\mathrm{Gr}(h) = \mathrm{Gr}(f) \cup \mathrm{Gr}(g)$. **[6 points]**

3. Use the Euclidean algorithm to compute the greatest common divisors of the following pairs of integers:

   $$(42, 30) \qquad (363, 154) \qquad (8085, 3094)$$

   Include the steps of the computation in your written solution. **[9 points]**

4. For each $n \in \mathbb{Z}$, let $D_n \subseteq \mathbb{Z}$ be the set of divisors of n. Prove that $D_a \cap D_b = D_{\mathrm{gcd}(a,b)}$ for all $a, b \in \mathbb{Z}$. **[7 points]**

5. Let $a, b \in \mathbb{Z}$ and suppose that d_1 and d_2 are greatest common divisors of a and b. Prove that either $d_1 = d_2$ or $d_1 = -d_2$. **[7 points]**