Functions and cardinality (advanced questions)

21-127 sections A and F TA: Clive Newstead 29th April 2014

What follows is a list of questions that you might want to try in preparation for my review session on functions on cardinality, which is

Tuesday 6th May 2014 at 7–10pm in Wean Hall 8220

These questions range from being realistic but tough, to outright impossible.

A1 A function $g : A \to B$ is *inflationary* if $g(x) > x$ for all $x \in A$ (where $A, B \subseteq \mathbb{R}$).

Prove that there exists an inflationary bijection $g : \mathbb{Z} \to \mathbb{Z}$, but that there does not exist an inflationary bijection $\mathbb{N} \to \mathbb{N}$. Does there exist an inflationary bijection $\mathbb{Z} \to \mathbb{N}$?

A2 A function $h : \mathbb{Z} \to \mathbb{Z}$ is *periodic* if there exists $m \in \mathbb{N}$ such that $\forall x \in \mathbb{Z}.\ h(x+m) = h(x)$. Prove that the set of periodic functions $\mathbb{Z} \to \mathbb{Z}$ is countable.

A3 Let Σ be a countably infinite set. Let Σ^* be the set of finite strings whose symbols come from Σ, and Σ^∞ be the set of infinite strings whose symbols come from Σ. Prove that Σ^* is countable but Σ^∞ is uncountable.

A4 A real number x is *algebraic* if x is a root of a polynomial with integer coefficients, i.e. if there exists $k \in \mathbb{N}$ and integers $a_0, a_1, \cdots, a_k$ such that

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k = 0$$

Prove that the set of algebraic real numbers is countably infinite.

A5 Let X be a set such that, for all $f : X \to X$, the following holds:

$$\forall x \in X.\ \exists n \in \mathbb{N}.\ f^n(x) = x \quad \Rightarrow \quad \exists n \in \mathbb{N}.\ \forall x \in X.\ f^n(x) = x$$

where $f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$. Prove that X is finite.

A6 Let S be a collection of pairwise disjoint intervals of $\mathbb{R}$ of positive length. That is,

$$S = \{(a_i, b_i) \subseteq \mathbb{R} : i \in I\}$$

with $a_i < b_i$ for all i, and if $i \neq j$ then $(a_i, b_i) \cap (a_j, b_j) = \varnothing$. Prove that S is countable. *(For clarity, $(a, b) = \{x \in \mathbb{R} : a < x < b\}$, not the ordered pair.)*

A7 The *successor* of a set x is the set $x^+ = x \cup \{x\}$. Define $\bar{n}$ for $n \in \mathbb{N} \cup \{0\}$ as follows:

$$\overline{0} = \varnothing \quad \text{and} \quad \overline{n+1} = \overline{n}^+ \text{ for all } n \in \mathbb{N} \cup \{0\}$$

For example, $\bar{1} = \varnothing \cup \{\varnothing\} = \{\varnothing\}$ and $\bar{2} = \bar{1} \cup \{\bar{1}\} = \{\varnothing, \{\varnothing\}\}$. Prove that $|\bar{n}| = n$ for all $n \in \mathbb{N} \cup \{0\}$.

A8 Does there exist a set S such that $|\mathbb{N}| < |S| < |\mathcal{P}(\mathbb{N})|$? *(Don't spend too much time on this.)*