# 21-128 problem sheet 4

Solutions to starred (*) exercises are due at the beginning of recitation on

**Thursday 8th October 2015**

Please submit answers to separate questions on separate sheets of paper.

**Problem 1 — 4.1**

Let $120102_{(3)}$ and $110222_{(3)}$ be ternary representations of two natural numbers. Use base 3 arithmetic to add them. Check the answer by converting each to base 10, adding, and converting back to base 3.

**Problem 2 — 4.2**

Which integer is bigger, $333_{(12)}$ or $3333_{(5)}$?

**Problem 3 — 4.3 ***

Note that $15^2 = 225$, $25^2 = 625$, $35^2 = 1225$. For $n \in \mathbb{N}$, prove that the square of the number given by appending 5 to the base 10 representation of $n$ is the number given by appending 25 to the base 10 representation of $n(n+1)$.

**Problem 4 — 4.6**

Let $A$ be the set of days in the week. Let $f$ assign to each day the number of letters in its English name. Does $f$ define an injection from $A$ to $\mathbb{N}$?

**Problem 5 — 4.8**

Let $f$ and $g$ be polynomials defined by $f(x) = x - 1$ and $g(x) = x^2 - 1$ for all $x \in \mathbb{R}$. Find formulae for $f \circ g$ and $g \circ f$.

**Problem 6 — 4.10 ***

Let $a, b, c, d \in \mathbb{R}$ be constants with $a \neq 0$ and $c \neq 0$. Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ with $f(x) = ax+b$ and $g(x) = cx + d$ for all $x \in \mathbb{R}$. Prove that $f$ and $g$ are injective and surjective, but that the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h = g \circ f - f \circ g$ is neither injective nor surjective.

**Problem 7 — 4.13 ***

Given a three-digit decimal number $n$, written as $\alpha\beta\gamma$ in base 10, the *reverse* of $n$ is the number written as $\gamma\beta\alpha$ in base 10. (For example, the reverse of 235 is 532, and the reverse of 3 is 300.)

Let $n$ be an integer between 1 and 999 written as $abc$ in base 10. Suppose $a \neq c$. Let $x$ be the (positive) difference between $n$ and its reverse. Prove that the sum of $x$ and its reverse is 1089.

**Problem 8 — 4.15 ***

Consider a balance scale plus $k$ objects of know weights $1, 3, \ldots, 3^{k-1}$. PRove by induction on $k$ that every unknown weight in the set $\left\{1, 2, \ldots, \frac{3^k-1}{2}\right\}$ can be balanced.

**Problem 9 — 4.17 ***

A position in the game of *Nim* consists of some piles of coins. Two players alternate, with each move removing a portion of one pile. The winner is the player who takes the last coin.

Suppose that the starting piles have sizes $n_1, \ldots, n_k$. Prove that Player 2 has a winning strategy if and only if, for every $j$, an even number of $n_1, \ldots, n_k$ have a 1 in position $j$ in their binary representation. For example, when the sizes are $1, 2, 3$, the binary representations are $1, 10, 11$, and the condition holds.

**Problem 10 — 4.24 ***

Let $f$ and $g$ be surjections from $\mathbb{Z}$ to $\mathbb{Z}$, and let $h = fg$ be their product (see Definition 1.25 in the book). Must $h$ also be surjective? Give a proof or a counterexample.

**Problem 11 — 4.25 ***

Determine which formulae below define surjections from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.

(a) $f(a, b) = a + b$;

(b) $f(a, b) = ab$;

(c) $f(a, b) = \frac{ab(b+1)}{2}$;

(d) $f(a, b) = \frac{(a+1)a(b+1)}{2}$;

(e) $f(a, b) = \frac{ab(a+b)}{2}$.