# 21-128 and 15-151 problem sheet 2

Solutions to the following seven exercises and optional bonus problem are to be submitted through blackboard by 8:30AM on

**Thursday 15th September 2016.**

There are also some practice problems, not to be turned in, for those seeking more practice and also for review prior to the exam.

### Problem 1

For what conditions on sets $A$ and $B$ does $A \setminus B = B \setminus A$ hold?

### Problem 2

Assuming only arithmetic (not the quadratic formula or calculus), prove that

$$\{x \in \mathbb{R} : x^2 - 2x - 3 < 0\} = \{x \in \mathbb{R} : -1 < x < 3\}$$

### Problem 3

Let $S = [3] \times [3]$ (the Cartesian product of $\{1, 2, 3\}$ with itself). Let $T$ be the set of ordered pairs $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ such that $0 \leq 3x + y - 4 \leq 8$. Prove that $S \subseteq T$. Does equality hold?

### Problem 4

Prove the following identities involving complementation of sets.

(a) $(A \cup B)^c = A^c \cap B^c$;

(b) $A \cap [(A \cap B)^c] = A \setminus B$;

(c) $A \cap [(A \cap B^c)^c] = A \cap B$;

(d) $(A \cup B) \cap A^c = B \setminus A$.

## Problem 5

Let $f : A \to B$ be a function and let $C, D \subseteq A$.

(a) Prove that $f(C \cap D) \subseteq f(C) \cap f(D)$.

(b) Give an example to demonstrate that equality need not hold in (a).

## Problem 6

Let $\mathbb{N}^+$ denote the set of positive integers and consider the function $f : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{R}$ defined by

$$f(a,b) = \frac{(a+1)(a+2b)}{2}$$

(a) Show that the image of $f$ is a subset of $\mathbb{N}^+$.

(b) Determine exactly which positive integers are elements of the image of $f$. (**Hint:** Formulate a hypothesis by trying values.)

## Problem 7

For $a \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, show that (a) and (b) below have different meanings.

(a) $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})[|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon]$.

(b) $(\exists \delta > 0)(\forall \varepsilon > 0)(\forall x \in \mathbb{R})[|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon]$.

(**Hint:** Find a function $f : \mathbb{R} \to \mathbb{R}$ and an element $a \in \mathbb{R}$ for which (a) and (b) have different truth values.)

## Bonus Problem (2 points)

Let $S$ be a non-empty set of people in a bar. Express the following statement symbolically:

> *There is a person in the bar such that, if that person is drinking, then everyone else in the bar is drinking.*

Prove that it is true.

**Extra Problem 1**

Let $A = \{\text{January}, \text{February}, \dots, \text{December}\}$. Given $x \in A$, let $f(x)$ be the number of days in $x$. Does $f$ define a function from $A$ to $\mathbb{N}$?

**Extra Problem 2**

Let $A, B \subseteq \mathbb{R}$, let $f : \mathbb{R} \to \mathbb{R}$ and let $P$ denote the set of positive real numbers. Without using negation words (e.g. 'not'), negate the following expressions:

(a) For all $x \in A$, there is a $b \in B$ such that $b > x$.

(b) There is an $x \in A$ such that, for all $b \in B$, $b > x$.

(c) For all $x, y \in \mathbb{R}$, if $f(x) = f(y)$ then $x = y$.

(d) For all $b \in \mathbb{R}$, there is an $x \in \mathbb{R}$ such that $f(x) = b$.

(e) For all $x, y \in \mathbb{R}$ and all $\varepsilon \in P$, there is a $\delta \in P$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

(f) For all $\varepsilon \in P$, there is a $\delta \in P$ such that, for all $x, y \in \mathbb{R}$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

**Extra Problem 3**

A function $g : \mathbb{R} \to \mathbb{R}$ is *even* if $g(-x) = g(x)$ for all $x \in \mathbb{R}$, or *odd* if $h(-x) = -h(x)$ for all $x \in \mathbb{R}$.

Let $f : \mathbb{R} \to \mathbb{R}$.

(a) Prove that there exists a unique pair of functions $g, h : \mathbb{R} \to \mathbb{R}$ such that $g$ is even, $h$ is odd, and $f = g + h$. (**Hint:** Express both $f(x)$ and $f(-x)$ in terms of $g(x)$ and $h(x)$, and solve the resulting system of equations.)

(b) When $f$ is a polynomial function, express $g$ and $h$ as in (a) in terms of the coefficients of $f$.