21-128 and 15-151 problem sheet 6

Solutions to the following seven exercises and optional bonus problem are to be submitted through blackboard by 8:30AM on

Thursday 20th October 2016.

There are also some practice problems, not to be turned in, for those seeking more practice and also for review prior to the exam.

Problem 1

Given a three-digit decimal number n, written as $\alpha\beta\gamma$ in base 10, the *reverse* of n is the number written as $\gamma\beta\alpha$ in base 10. (For example, the reverse of 235 is 532, and the reverse of 3 is 300.)

Let n be an integer between 1 and 999 written as abc in base 10. Suppose $a \neq c$. Let x be the (positive) difference between n and its reverse. Prove that the sum of x and its reverse is 1089.

Problem 2

Let $a, b, c, d \in \mathbb{R}$ be constants with $a \neq 0$ and $c \neq 0$. Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ with $f(x) = ax+b$ and $g(x) = cx + d$ for all $x \in \mathbb{R}$. Prove that f and g are injective and surjective, but that the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h = g \circ f - f \circ g$ is neither injective nor surjective.

Problem 3

Verify that the function $f : (0, 1) \to \mathbb{R}$ defined by

$$f(x) = \frac{2x - 1}{2x(1 - x)} \quad \text{for all } x \in (0, 1)$$

is a bijection.

Problem 4

Consider three functions $f, g, h : \mathbb{R} \to \mathbb{R}$, defined for all $x \in \mathbb{R}$ by

$$f(x) = \frac{x}{1+x^2}, \quad g(x) = \frac{x^2}{1+x^2}, \quad h(x) = \frac{x^3}{1+x^2}$$

(a) Determine which of these functions are injective.

(b) Prove that f and g are not surjective.

Problem 5

Given real numbers a, b, c, d, let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (ax + by, cx + dy)$ for all $(x, y) \in \mathbb{R}^2$. Prove that f is injective if and only if f is surjective.

Problem 6

Let $f : A \to B$ and $g : B \to C$ be functions, and define $h = g \circ f$. Determine which of the following statements are true, giving proofs for the true statements and counterexamples for the false statements:

(a) If h is injective, then f is injective.

(b) If h is injective, then g is injective.

(c) If h is surjective, then f is surjective.

(d) If h is surjective, then g is surjective.

Problem 7

Consider functions $f : A \to B$ and $g : B \to A$. Prove that

(a) If $f \circ g$ is the identity function on B, then f is surjective.

(b) If $g \circ f$ is the identity function on A, then f is injective.

To remind you: given a set X, the identity function on X is the function $\mathrm{id}_X : X \to X$ defined by $\mathrm{id}_X(x) = x$ for all $x \in X$.

Bonus Problem - 2 points

If the common difference d of an arithmetic progression starting from 1 is relatively prime to 10, show that the sequence $1,\ 1+d,\ 1+2d,\ \ldots$ contains infinitely many powers of 10. Is this still true if the arithmetic progression starts from 2?

Extra Problem 1

Consider a function $f : A \to A$. Prove that if $f \circ f$ is injective, then f is injective.

Extra Problem 2

Let $f : A \to B$ be a function.

(a) Prove that there exists a set X and functions $p : A \to X$ and $i : X \to B$, with p surjective and i injective, such that $f = i \circ p$.

(b) Prove that there exists a set Y and functions $j : A \to Y$ and $q : Y \to B$, with j injective and q surjective, such that $f = q \circ j$.

Extra Problem 3

Recall that if $A, B \subseteq \mathbb{R}$, a function $f : A \to B$ is **increasing** if, for all $x, y \in A$, if $x < y$ then $f(x) < f(y)$.

Let A and B be subsets of $\mathbb{R}$ and let $f : A \to B$ be a bijection. Prove that if f is increasing, then f^{-1} is increasing.