21-256: Implicit partial differentiation

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Introduction

This note is a slightly different treatment of implicit partial differentiation from what I did in class and follows more closely what I *wanted* to say to you. I'm doing this with the hope that the third iteration will be clearer than the first two!

We say variables x, y, z are *related implicitly* if they depend on each other by an equation of the form $F(x, y, z) = 0$, where F is some function. For example, the points on a sphere centred at the origin with radius 3 are related by the equation $x^2 + y^2 + z^2 - 9 = 0$. In such situations, we may wish to know how to compute the partial derivatives of one of the variables with respect to the other variables.

To do so, we have to do something quite subtle. On one hand, we want to treat the variables as *independent* in order to find the partial derivatives of the function F. On the other hand, we want to take into account the *dependence* of the variables on one another, via the equation $F(x, y, z) = 0$.

Why the chain rule is appropriate

The chain rule says that if F is a function of 'old' variables x, y, z, each of which is a function of 'new' variables s, t, then

$$\frac{\partial F}{\partial s} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial s}$$

The mind-warping element of implicit differentiation is that we're going to take s and t to be two of the three of x, y, z. (Doing so is not just a cosmetic difference, it affects the differentiation!)

Indeed, the 'old' variables will be x, y, z treated as if they're independent, from the perspective of someone looking at the function without the information that $F(x, y, z) = 0$. The 'new' variables will be whatever's left after we choose to consider one as a function of the other two, in such a way that the equation $F(x, y, z) = 0$ is satisfied.

Hence the $\dfrac{\partial F}{\partial s}$ above will always be zero, because after we take the dependence $F(x, y, z)$ into account, we're left with a constantly zero function!

How it's done

Suppose x, y, z are related by $F(x, y, z) = 0$ and you want to compute $\dfrac{\partial z}{\partial x}$. To do this, we treat z as if it were a function of x and y. Thus, in the language of the chain rule:

- The 'old' variables are x, y, z, treated as being completely independent;
- The 'new' variables are just x and y, because we want to treat z as a function of x and y.

In the chain rule, when we differentiate with respect to the 'old' variables we do so as if they had no dependence on the 'new' variables. To distinguish between 'old' vs 'new', write $s = x$ and $t = y$ when they're considered as new variables. Thus

$$\frac{\partial F}{\partial s} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial s}$$

Now $x = s$, so $\frac{\partial x}{\partial s} = 1$. And $y = t$, so $\frac{\partial y}{\partial s} = 0$. So really we have

$$\frac{\partial F}{\partial s} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial s}$$

We can then write s as x once again to obtain

$$\frac{\partial z}{\partial s} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

This result is known as the **implicit function theorem**.

Example

Suppose x, y, z are variables related by the equation $x^4 + y^4 + z^4 + x^2y^2z^2 = 0$, and that we want to find $\dfrac{\partial y}{\partial z}$.

We thus treat y as a function of x and z. So the 'old' variables are x, y, z and the 'new' variables are x, z, with

$$\frac{\partial x}{\partial z} = 0 \qquad \text{and} \qquad \frac{\partial z}{\partial z} = 1$$

Write $F(x, y, z) = x^4 + y^4 + z^4 + x^2y^2z^2$ for the function where x, y, z are considered unrelated. Then

$$\frac{\partial F}{\partial z} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial z}$$

where on the LHS F is the function in which y depends on x and z via $F(x, y, z) = 0$, and on the RHS the Fs treat their variables as independent. (If your reaction is "what?!" then see the bottom the next page.)

Thus

$$0 = (\text{stuff}) \cdot 0 + (4y^3 + 2x^2yz^2) \cdot \frac{\partial y}{\partial x} + (4z^3 + 2x^2y^2z) \cdot 1$$

and hence

$$\frac{\partial y}{\partial x} = -\frac{4z^3 + 2x^2y^2z}{4y^3 + 2x^2yz^2} = -\frac{z(2z^2 + x^2y^2)}{y(2y^2 + x^2z^2)}$$

(What?! Don't panic, this is just what we did before. Really we should have written $x = s$, $z = t$ and $y = y(s, t)$ so that $F(s, y(s, t), t) = 0$, giving

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial t}$$

and then relabelled $s \to x$ and $t \to y$.)

Example (with less waffle)

Suppose $x^2y^2 + y^2z^2 + z^2x^2 = 0$ and we want to compute $\dfrac{\partial x}{\partial y}$.

- Write $F(x, y, z) = x^2y^2 + y^2z^2 + z^2x^2$.
- Treat x as a function of y, z and y, z as independent; thus $\frac{\partial y}{\partial y} = 1$ and $\frac{\partial z}{\partial y} = 0$.
- Differentiate the equation $F(x, y, z) = 0$ with respect to y; we get

$$\begin{aligned} 0 = \frac{\partial F}{\partial y} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} \\ &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial F}{\partial y} \cdot 1 + \frac{\partial F}{\partial z} \cdot 0 \\ &= (2xy^2 + 2xz^2)\frac{\partial x}{\partial y} + (2x^2y + 2yz^2) + (\text{stuff}) \cdot 0 \end{aligned}$$

  Hence

$$\frac{\partial x}{\partial y} = -\frac{2x^2y + 2yz^2}{2xy^2 + 2xz^2} = -\frac{y(x^2 + z^2)}{x(y^2 + z^2)}$$

On the implicit function theorem

We could have just used the implicit function theorem; if you do so on your homework, please at least calculate the first partial derivatives of the function F.