21-256: Integration

Clive Newstead, Monday 23rd June 2014

Recall that if f is a function of a single variable then $\displaystyle\int_a^b f(x)\,dx$ is equal to the area of the region under the curve $y = f(x)$ lying above the x-axis between $x = a$ and $x = b$ (minus the area lying below the x-axis). Bumping up the dimension, we can thus ask: given a region U of the xy-plane, what is the volume under a surface $z = f(x, y)$ lying above U?

We denote this volume by $\displaystyle\int_U f(x,y)\,dA$. (The dA represents an infinitesimal quantity of area.)

Integration over rectangular regions of $\mathbb{R}^2$

If U is a rectangular region of the form $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$, then

$$\int_U f(x,y)\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

The two integrals on the right-hand side are *partial integrals*; the inner integral is with respect to y, treating x as constant. Thus $\displaystyle\int_c^d f(x,y)\,dy$ is a function of x.

The following all denote the same thing:

$$\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx, \qquad \int_a^b \int_c^d f(x,y)\,dy\,dx, \qquad \int_a^b dx \int_c^d dy\, f(x,y)$$

We'll use the middle one.

Theorem (Fubini). If f is suitably nice (namely, continuous on the rectangle $a \leqslant x \leqslant b$ and $c \leqslant y \leqslant d$), then

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

That is, when the limits are constant, the order of integration doesn't matter.

Integration over more general regions of $\mathbb{R}^2$

If U can be described as $a \leqslant x \leqslant b$ and $g_1(x) \leqslant y \leqslant g_2(x)$, then

$$\int_U f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

If U can be described as $c \leqslant y \leqslant d$ and $h_1(y) \leqslant x \leqslant h_2(y)$, then

$$\int_U f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

This is especially useful if we want to integrate over a region bounded by two curves.

Not all regions are this simple, but often they can be subdivided into regions that *can* be described in this way.

Definition. Let U be a subset of $\mathbb{R}^2$. A (*finite*) *partition* of U is a subdivision of U into smaller sets $U_1, U_2, \cdots, U_n$ such that the 'U_i's don't overlap, except possibly on their boundaries.

Theorem. If $U_1, U_2, \cdots, U_n$ is a partition of U, then

$$\int_U f(x,y)\,dA = \sum_{i=1}^{n} \int_{U_i} f(x,y)\,dA$$

Average values

In one dimension, the average value of $f(x)$ on the interval $a \leqslant x \leqslant b$ is given by

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

Notice that $b - a$ is the length of the interval. The two-dimensional analogue of length is area; and sure enough, we use area to define average values.

Let f be a function of two variables x, y and let U be a subset of the xy-plane. The *average value* of $f(x,y)$ on U is

$$\frac{1}{A(U)}\int_U f(x,y)\,dA$$

where $A(U)$ is the area of U.