21-256: Lines and planes

Clive Newstead, Friday 30th May 2014

This is a summary of the important results about lines and planes that you should know.

Lines

A line in $\mathbb{R}^n$ is determined by a point A on the line and a direction $\mathbf{v}$ in which the line points. Writing $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{r} = \overrightarrow{OP}$, we have that P lies on the line if and only if

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{v}$$

This is called the *vector equation* of the line.

Two lines are parallel if and only if their direction vectors are parallel or opposite; that is the lines

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{v} \qquad \text{and} \qquad \mathbf{r} = \mathbf{b} + \mu \mathbf{w}$$

are parallel if and only if $\mathbf{v} = k\mathbf{w}$ for some scalar $k \neq 0$.

If there is a point lying on two lines then they *intersect*, otherwise they are *skew*. We can determine whether two lines intersect by trying to solve the equation $\mathbf{a} + \lambda \mathbf{v} = \mathbf{b} + \mu \mathbf{w}$ for λ and μ. If a solution exists then they intersect, and the point at which they intersect can be found by plugging λ into the equation of the first line and μ into the equation of the second line (you'll get the same answer). If no solution exists, the lines are skew.

Planes

A plane in $\mathbb{R}^3$ is determined by a point A on the plane and a direction $\mathbf{n}$ which is perpendicular to the plane, called the *normal vector* to the plane. Writing $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{r} = \overrightarrow{OP}$, we have that P lies on the plane if and only if

$$\mathbf{n} \cdot \overrightarrow{AP} = 0 \qquad \text{i.e.} \quad \mathbf{n} \cdot (\mathbf{r} - \mathbf{a}) = 0$$

This is called the *vector equation* of the plane.

Writing this out, say $\mathbf{n} = p\mathbf{i} + q\mathbf{j} + r\mathbf{k}$ and $A = (a, b, c)$, we thus have that (x, y, z) lies on the plane if and only if

$$p(x - a) + q(y - b) + r(z - c) = 0 \quad \text{i.e.} \quad px + qy + rz = s$$

where $s = pa + qb + rc$. This is called the *linear equation* of the plane.

Intersections and angles

Two planes $\mathbf{n}_1 \cdot (\mathbf{r} - \mathbf{a}_1) = 0$ and $\mathbf{n}_1 \cdot (\mathbf{r} - \mathbf{a}_2) = 0$ are *parallel* if and only if their normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel or opposite. If two planes are parallel then they either intersect everywhere (they're the same plane) or they don't intersect at all.

If two planes are not parallel then they intersect at a line. The equation of their line of intersection can be solved by finding

- Point on line: to do this, solve the linear equations of the planes simultaneously.
- Direction: this must be perpendicular to both $\mathbf{n}_1$ and $\mathbf{n_2}$, so the line has direction $\mathbf{n}_1 \times \mathbf{n_2}$.

Basic geometry shows that the angle between two planes is equal to the angle between their normal vectors. Thus the angle between planes $\mathbf{n}_1 \cdot (\mathbf{r} - \mathbf{a}_1) = 0$ and $\mathbf{n}_1 \cdot (\mathbf{r} - \mathbf{a}_2) = 0$ is given by

$$\cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|}\right)$$

Distance from a point to a plane

If P is a point, then the distance from P to the plane containing a point A and perpendicular to a vector $\mathbf{n}$ is given by $|\mathrm{comp}_{\mathbf{n}}(\overrightarrow{AP})|$. Writing $\mathbf{p} = \overrightarrow{OP}$ and $\mathbf{a} = \overrightarrow{OA}$, we see that the distance is thus

$$\left|\frac{\mathbf{n} \cdot (\mathbf{p} - \mathbf{a})}{\|\mathbf{n}\|}\right|$$

Linear dependence and independence

A *linear combination* of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a vector of the form

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v_2} + \cdots + \lambda_n \mathbf{v}_n$$

The linear combination is *nontrivial* if at least one of the λ_is is nonzero, or *trivial* otherwise.

Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are *linearly dependent* (LD) if the zero vector is a nontrivial linear combination of them. Otherwise they are *linearly independent* (LI). Explicitly:

- $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are LD if there exist $\lambda_1, \lambda_2, \ldots, \lambda_n$ not all zero such that $\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v_2} + \cdots + \lambda_n \mathbf{v}_n = \mathbf{0}$.
- $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are LI if $\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n = \mathbf{0}$ implies that $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.