Math 290-1 Class 3

C.H.: On October 3rd, he asked me what day it was.
K.S.: On Wednesdays we wear pink.

—Mean Girls (2004)

A matrix is in **reduced row echelon form** (**rref**) if:

(1) If a row has a nonzero entry, the first nonzero entry is a 1 (called a pivot).

(2) If a column contains a pivot, **all other entries** in that column are zero.
If just entries below the pivot are zero: just 'row echelon form' (ref) — still useful for solving systems

(3) If a row contains a pivot, each row above it contains a pivot.

When you have put the augmented matrix in rref, each equation will contain one (and only one) leading variable. This means the solution can be read off from the augmented matrix with ease.

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 3 & 4 & -6 & 8 & 0 \\ 0 & -1 & 3 & 4 & -12 \end{array}\right) \overset{\text{rref}}{\rightsquigarrow} \left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \rightsquigarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2t \\ 4+3t \\ t \\ -2 \end{pmatrix}$$

What follows is a failsafe algorithm for putting a matrix into reduced row-echelon form.

- **Check for pivots.** If all leading nonzero entries are 1s and are the only nonzero entry in their column, go to *'Swap the rows'* step below. Otherwise, proceed to the next step.
- **Pick a pivot.** Identify a nonzero entry with only zeroes to the left of it. This will be our choice of pivot.
- **Divide through.** If the entry is not 1, divide that row by the entry to make it become 1.
- **Clear the column.** Subtract multiples of that row from the other rows to clear all nonzero entries in that column.
- **Repeat.** Go back to the *'Check for potential pivots'* step.
- **Swap the rows.** Permute the rows so that all zero rows to the bottom and put the pivots in the correct order. Congratulations! Your matrix is now in rref.

More rref fun facts

- Every matrix A has one (and only one) reduced row-echelon form, written $\mathrm{rref}(A)$.
- If A can be transformed into B using elementary row operations, then $\mathrm{rref}(A) = \mathrm{rref}(B)$.
- If $\mathrm{rref}(A) = \mathrm{rref}(B)$, then A can be transformed into B using elementary row operations. (The last fact is pretty surprising, if you think about it!)

Rank and the number of solutions

The **coeffient matrix** of a linear system is the matrix to the left of the vertical bar in the augmented matrix—it contains the coefficients of the variables, but not the constants.

$$\begin{cases} x + 2y + 3z = 4 \\ 4x + 5y + 6z = 7 \end{cases} \quad \rightsquigarrow \quad \overbrace{\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \end{array}\right)}^{\text{augmented matrix}}$$

with the coefficient matrix $\underbrace{\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{matrix}}_{\text{coefficient matrix}}$

- The number of **rows** in the coefficient matrix is the number of equations in the system;
- The number of **columns** in the coefficient matrix is the number of variables in the system.

The **rank** of a matrix A is the number of pivots in $\mathrm{rref}(A)$.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad \rightsquigarrow \quad \mathrm{rref}(A) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} \quad \rightsquigarrow \quad \mathrm{rank}(A) = 2$$

Observations:

- If A is an $m \times n$ matrix with rank ρ, then $\rho \leqslant \max\{m, n\}$.
- Row echelon form works just as well as *reduced* row echelon form for computing the rank.

Comparing the rank with the number of rows and columns gives us information about the number of solutions to the system. The following table indicates how many solutions a system of equations whose coefficient matrix has rank ρ might have.

	$\rho =$ # columns	$\rho <$ # columns
$\rho =$ # rows	1 solution	∞ solutions
$\rho <$ # rows	0 or 1 solutions	0 or ∞ solutions

Advice: Don't memorise this table! Use intuition in 2 and/or 3 dimensions to reconstruct it.

Looking at the rref of the *augmented* matrix allows us to determine whether the system is consistent.

1. [Bretscher §1.2 Q12, modified] Find the rank of the following matrix.

$$\begin{pmatrix} 2 & 0 & -3 & 0 & 7 & 7 \\ -2 & 1 & 6 & 0 & -6 & -12 \\ 0 & 1 & -3 & 0 & 1 & 5 \\ 0 & -2 & 0 & 1 & 1 & 1 \\ 2 & 1 & -3 & 0 & 8 & 7 \end{pmatrix}$$

2. For each of the following statements, determine whether it is *always* true, *sometimes* true, or *never* true.

(a) A linear system with more rows than variables has no solution.

always sometimes never

(b) A linear system with more variables than rows has a unique solution.

always sometimes never

(c) If there are n equations in a linear system and its coefficient matrix has rank n, then it has a unique solution.

always sometimes never

(d) If there are n variables in a linear system and its coefficient matrix has rank n, then it has a unique solution.

always sometimes never