1. [Bretscher §1.2 Q12, modified] Find the rank of the following matrix.

$$\begin{pmatrix} 2 & 0 & -3 & 0 & 7 & 7 \\ -2 & 1 & 6 & 0 & -6 & -12 \\ 0 & 1 & -3 & 0 & 1 & 5 \\ 0 & -2 & 0 & 1 & 1 & 1 \\ 2 & 1 & -3 & 0 & 8 & 7 \end{pmatrix}$$

Use this 2 as an "almost pivot" — we'll clear the other nonzero entries in column 1.

$$\begin{array}{cl} \begin{pmatrix} 2 & 0 & -3 & 0 & 7 & 7 \\ 0 & 1 & 3 & 0 & 1 & -5 \\ 0 & 1 & -3 & 0 & 1 & 5 \\ 0 & -2 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} & \begin{array}{l} \\ (\mathrm{II})+(\mathrm{I}) \\ \\ \\ (\mathrm{V})+(\mathrm{I}) \end{array} \end{array}$$

Use this 1 to clear the nonzero entries below it.

$$\begin{array}{cl} \begin{pmatrix} 2 & 0 & -3 & 0 & 7 & 7 \\ 0 & 1 & 3 & 0 & 1 & -5 \\ 0 & 0 & -6 & 0 & 0 & 10 \\ 0 & 0 & 6 & 1 & 3 & -9 \\ 0 & 0 & -3 & 0 & 0 & 5 \end{pmatrix} & \begin{array}{l} \\ \\ (\mathrm{III})-(\mathrm{II}) \\ (\mathrm{IV})+2\times(\mathrm{II}) \\ (\mathrm{V})-(\mathrm{II}) \end{array} \end{array}$$

Again we'll use this −3 as an "almost pivot" to clear the −6 and 6

$$\begin{array}{cl} \begin{pmatrix} 2 & 0 & -3 & 0 & 7 & 7 \\ 0 & 1 & 3 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & -3 & 0 & 0 & 5 \end{pmatrix} & \begin{array}{l} \\ \\ (\mathrm{III})-2\times(\mathrm{V}) \\ (\mathrm{IV})+2\times(\mathrm{V}) \\ \\ \end{array} \end{array}$$

Finally move the row of 0s to the bottom & put leading nonzero entries in order

$$\begin{array}{cl} \begin{pmatrix} 2 & 0 & -3 & 0 & 7 & 7 \\ 0 & 1 & 3 & 0 & 1 & -5 \\ 0 & 0 & -3 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \text{swap} \end{array}$$

The matrix has rank 4 because there are four leading nonzero entries.

Remarks

① Dividing (I) by 2 and (III) by −3 would put the matrix in row echelon form. Then adding / subtracting appropriate multiples of (III) from (I), (II) to clear the −3 and 3 would put it in reduced row echelon form.

② The additional work I just mentioned was not needed to find the rank!

2. For each of the following statements, determine whether it is *always* true, *sometimes* true, or *never* true.

(a) A linear system with more rows than variables has no solution.

always **sometimes** never

eg/ $\begin{cases} x = 1 \\ y = 1 \\ x + y = 0 \end{cases}$ has no solution but $\begin{cases} x = 1 \\ y = 1 \\ x + y = 2 \end{cases}$ has 1 solution : $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(b) A linear system with more variables than rows has a unique solution.

always sometimes **never**

The rank (= # leading variables) is $\leq$ the # equations

$\Rightarrow$ # variables > # equations $\geq$ # leading variables

$\Rightarrow$ # free variables = # variables − # leading variables > 0

This means that any solution will have parameters, so there cannot be a unique solution.

(c) If there are n equations in a linear system and its coefficient matrix has rank n, then it has a unique solution.

always **sometimes** never

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

eg/ $\begin{cases} x = 1 \\ y = 1 \end{cases}$ has a unique solution $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

but $\begin{cases} x + z = 1 \\ y + z = 1 \end{cases}$ has ∞'ly many solutions $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1-t \\ 1-t \end{pmatrix}$ for all t.

(d) If there are n variables in a linear system and its coefficient matrix has rank n, then it has a unique solution.

always **sometimes** never

eg/ Same examples as in part (a) : in both examples, the coefficient matrix is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$, which has rank 2 since its rref is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ — it is possible that there is no solution.