Math 290-1 Class 4

Friday 5th October 2018

Vector algebra

- **Vector sum.** If $\vec{v}$ and $\vec{w}$ are n-dimensional vectors, then $\vec{v}+\vec{w}$ is the n-dimensional vector whose i^{th} component is $v_i + w_i$.

  $$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{pmatrix}$$

  Likewise for row vectors. (The sum of a row vector and a column vector is not defined.)

- **Scalar multiplication.** If $\vec{v}$ is an n-dimensional vector and k is a real number ('scalar'), then $k\vec{v}$ is the n-dimensional vector whose i^{th} component is kv_i.

  $$k\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} kv_1 \\ kv_2 \\ \vdots \\ kv_n \end{pmatrix}$$

  Likewise for row vectors.

- A **linear combination** of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is a vector of the form $k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n$, where $k_1, k_2, \dots, k_n$ are scalars.

- **Dot product.** If $\vec{v}$ and $\vec{w}$ are vectors with n components, their dot product is the scalar $\vec{v}\cdot\vec{w}$ defined by

  $$\vec{v}\cdot\vec{w} = v_1w_1 + v_2w_2 + \cdots + v_nw_n$$

  This is defined even if $\vec{v}$ and $\vec{w}$ are not both row vectors or both column vectors.

Matrix algebra

- **Matrix sum.** If A and B are $m \times n$ matrices, then $A+B$ is the $m \times n$ matrix whose (i,j) component is $a_{ij}+b_{ij}$. ($A+B$ is undefined if A and B have different sizes.)

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

- **Scalar multiplication.** If A is an $m \times n$ matrix and k is a real number ('scalar'), then kA is the $m \times n$ matrix whose (i,j) component is ka_{ij}.

$$k \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix}$$

- **Product of a matrix and a vector.** If A is an $m \times n$ matrix and $\vec{v}$ is an n-dimensional column vector, then $A\vec{v}$ is the m-dimensional column vector whose i^{th} component is $a_{i1}v_1 + a_{i2}v_2 + \cdots + a_{in}v_n$ for all $1 \leqslant i \leqslant m$.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \cdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{pmatrix}$$

  Notice that the $m \times n$ matrix A transforms vectors in $\mathbb{R}^n$ to vectors in $\mathbb{R}^m$ (we'll see much more of this on Monday).

  - ◇ Writing $\vec{a}_{\bullet i}$ for the m-dimensional column vector given by the i^{th} column of A, we have

$$A\vec{v} = \begin{pmatrix} \vdots & \vdots & & \vdots \\ \vec{a}_{\bullet 1} & \vec{a}_{\bullet 2} & \cdots & \vec{a}_{\bullet n} \\ \vdots & \vdots & & \vdots \end{pmatrix} \vec{v} = v_1\vec{a}_{\bullet 1} + v_2\vec{a}_{\bullet 2} + \cdots + v_n\vec{a}_{\bullet n}$$

  - ◇ Writing $\vec{a}_{j\bullet}$ for the n-dimensional row vector given by the j^{th} row of A, we have

$$A\vec{v} = \begin{pmatrix} \cdots & \vec{a}_{1\bullet} & \cdots \\ \cdots & \vec{a}_{2\bullet} & \cdots \\ & \vdots & \\ \cdots & \vec{a}_{m\bullet} & \cdots \end{pmatrix} \vec{v} = \begin{pmatrix} \vec{a}_{1\bullet} \cdot \vec{v} \\ \vec{a}_{2\bullet} \cdot \vec{v} \\ \vec{a}_{m\bullet} \cdot \vec{v} \end{pmatrix}$$

- Finally, notice that $A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w}$ and $A(k\vec{v}) = A\vec{v}$. (Keep this in mind for Monday—this tells us that A defines a *linear transformation*.)

1. For each of the following expressions, either compute its value, or explain why it is not defined.

(a) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}$

(c) $2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

(d) $2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

(f) $\begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & -1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$

2. Express the following linear system in the form $A\vec{x} = \vec{b}$.

$$\begin{cases} 2x_1 - x_2 + 3x_3 - x_4 + 2x_5 = 0 \\ x_3 + x_4 = 2 \\ -x_1 - x_2 + 5x_4 = -1 \\ 3x_1 - x_5 = 4 \end{cases}$$

[Your answer is called the **matrix form** of the system.]

3. Explain why every vector in $\mathbb{R}^3$ can be expressed as a linear combination of the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, where

$$\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$