Math 290-1 Class 6

Wednesday 10th October 2018

Linear transformations in geometry

We now focus on *square* matrices, which define linear transformations $T: \mathbb{R}^n \to \mathbb{R}^n$. (The fancy name for these are *linear endomorphisms* of \mathbb{R}^n .) Some examples of linear endomorphisms are: scaling, rotation, reflection through a line, and orthogonal projection onto a line.

[Note that translation is typically *not* a linear transformation, since it doesn't send $\vec{0}$ to $\vec{0}$.]

Recall from Monday that we can find the matrix of a linear transformation by considering its action on the standard basis vectors \vec{e}_i .

$$T(\vec{v}) = A\vec{v}$$
 where $A = \begin{pmatrix} \vdots & \vdots & & \vdots \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \\ \vdots & \vdots & & \vdots \end{pmatrix}$

For example, if $T: \mathbb{R}^2 \to \mathbb{R}^2$ is the matrix that rotates each point about the origin by an angle of θ , then

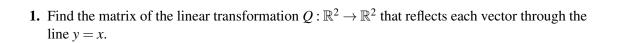
$$T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$
 and $T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$

so the matrix associated with T is $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$.

Orthogonal projection

Given a fixed vector \vec{a} , we can write any vector \vec{v} uniquely as $\vec{v}^{\parallel} + \vec{v}^{\perp}$, where \vec{v}^{\parallel} is parallel to \vec{a} and \vec{v}^{\perp} is perpendicular to \vec{a} .

The vector \vec{v}^{\parallel} is called the **orthogonal projection** of \vec{v} onto \vec{a} , and the assignment $\vec{v} \mapsto \vec{v}^{\parallel}$ defines a linear transformation $\operatorname{proj}_{\vec{a}} : \mathbb{R}^n \to \mathbb{R}^n$.



2. Find the matrix of the linear transformation $R : \mathbb{R}^3 \to \mathbb{R}^3$ that scales each vector by a factor of 2 and then reflects it in the (x,y)-plane.

3. Find the matrix of the linear transformation $S: \mathbb{R}^3 \to \mathbb{R}^3$ that rotates each vector by θ radians about the *y*-axis.

- **4.** Given a fixed nonzero vector \vec{a} , use the following three facts to find an explicit formula for the linear map $\operatorname{proj}_{\vec{a}} : \mathbb{R}^n \to \mathbb{R}^n$.
 - (i) $\vec{v}^{\parallel} = k\vec{a}$ for some scalar k (since \vec{v}^{\perp} is parallel to \vec{a})
 - (ii) $\vec{v}^{\perp} \cdot \vec{a} = 0$ (since \vec{v}^{\perp} is perpendicular to \vec{a})
 - (iii) $\vec{v} = \vec{v}^{\parallel} + \vec{v}^{\perp}$

[Hint: start by writing \vec{v}^{\perp} in terms of \vec{v} and \vec{v}^{\parallel} in equation (ii).]

Find the orthogonal projection of $\begin{pmatrix} 2\\3\\-1 \end{pmatrix}$ onto the vector $\begin{pmatrix} 1\\0\\1 \end{pmatrix}$.

5. Let ℓ be a line through the origin in \mathbb{R}^n which is parallel to a vector \vec{a} . Find an expression for the linear transformation $\operatorname{ref}_{\ell}: \mathbb{R}^n \to \mathbb{R}^n$ that reflects each vector \vec{v} through the line ℓ .

Find the result of reflecting the vector $\begin{pmatrix} 2\\3\\-1 \end{pmatrix}$ through the line which passes through the origin and is parallel to the vector $\begin{pmatrix} 1\\0\\1 \end{pmatrix}$.