Math 290-1 Class 6

Wednesday 10th October 2018

Linear transformations in geometry

We now focus on *square* matrices, which define linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$. (The fancy name for these are *linear endomorphisms* of $\mathbb{R}^n$.) Some examples of linear endomorphisms are: scaling, rotation, reflection through a line, and orthogonal projection onto a line.

[Note that translation is typically *not* a linear transformation, since it doesn't send $\vec{0}$ to $\vec{0}$.]

Recall from Monday that we can find the matrix of a linear transformation by considering its action on the standard basis vectors $\vec{e}_i$.

$$T(\vec{v}) = A\vec{v} \quad \text{where} \quad A = \begin{pmatrix} \vdots & \vdots & & \vdots \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

For example, if $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the matrix that rotates each point about the origin by an angle of θ, then

$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

so the matrix associated with T is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Orthogonal projection

Given a fixed vector $\vec{a}$, we can write any vector $\vec{v}$ uniquely as $\vec{v}^{\parallel} + \vec{v}^{\perp}$, where $\vec{v}^{\parallel}$ is parallel to $\vec{a}$ and $\vec{v}^{\perp}$ is perpendicular to $\vec{a}$.

The vector $\vec{v}^{\parallel}$ is called the **orthogonal projection** of $\vec{v}$ onto $\vec{a}$, and the assignment $\vec{v} \mapsto \vec{v}^{\parallel}$ defines a linear transformation $\text{proj}_{\vec{a}} : \mathbb{R}^n \to \mathbb{R}^n$.

1. Find the matrix of the linear transformation $Q : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects each vector through the line $y = x$.

2. Find the matrix of the linear transformation $R : \mathbb{R}^3 \to \mathbb{R}^3$ that scales each vector by a factor of 2 and then reflects it in the (x, y)-plane.

3. Find the matrix of the linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^3$ that rotates each vector by θ radians about the y-axis.

4. Given a fixed nonzero vector $\vec{a}$, use the following three facts to find an explicit formula for the linear map $\text{proj}_{\vec{a}} : \mathbb{R}^n \to \mathbb{R}^n$.

(i) $\vec{v}^{\parallel} = k\vec{a}$ for some scalar k (since $\vec{v}^{\perp}$ is parallel to $\vec{a}$)

(ii) $\vec{v}^{\perp} \cdot \vec{a} = 0$ (since $\vec{v}^{\perp}$ is perpendicular to $\vec{a}$)

(iii) $\vec{v} = \vec{v}^{\parallel} + \vec{v}^{\perp}$

[*Hint: start by writing $\vec{v}^{\perp}$ in terms of $\vec{v}$ and $\vec{v}^{\parallel}$ in equation (ii).*]

Find the orthogonal projection of $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ onto the vector $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

5. Let ℓ be a line through the origin in $\mathbb{R}^n$ which is parallel to a vector $\vec{a}$. Find an expression for the linear transformation $\mathrm{ref}_\ell : \mathbb{R}^n \to \mathbb{R}^n$ that reflects each vector $\vec{v}$ through the line ℓ.

Find the result of reflecting the vector $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ through the line which passes through the origin and is parallel to the vector $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.