Math 290-1 Class 9

Wednesday 17th October 2018

More inverse fun facts

- If A and B are $n \times n$ matrices and $AB = I_n$, then A and B are both invertible and $B = A^{-1}$ (and $A = B^{-1}$). This means that to verify that a matrix B is an inverse to a matrix A, all you have to do is check that $AB = I_n$.
- An $n \times n$ matrix A is invertible if and only if the only solution $\vec{x}$ to the equation $A\vec{x} = \vec{0}$ is the zero vector, $\vec{x} = \vec{0}$.
- Consequently, if $A\vec{v} = A\vec{w}$ for some $\vec{v} \neq \vec{w}$, then A is not invertible. (Why?)

Inverses and geometry

If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, its inverse $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is the linear transformation that reverses the action of T. For example:

- If T acts by rotation by an angle θ about some axis, then T^{-1} acts by rotation by θ in the opposite direction about the same axis. For example, in $\mathbb{R}^2$:
$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$
- If T acts by reflection, then $T^{-1} = T$. For example, in $\mathbb{R}^2$:
$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \qquad (a^2 + b^2 = 1)$$
- If T acts by scaling by a factor of $k > 0$, then T^{-1} acts by scaling by a factor of $\frac{1}{k}$. For example, in $\mathbb{R}^2$:
$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{k} & 0 \\ 0 & \frac{1}{k} \end{pmatrix}$$
- If T acts by orthogonal projection, then T is not invertible. (Why?)

The **determinant** of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the quantity $\det(A) = ad - bc$.

- The determinant tells us how size is scaled by $T(\vec{x}) = A\vec{x}$. The unit square maps under T to a (line or) parallelogram P, whose area is $|\det(A)|$.
- A is invertible if and only if $\det(A) \neq 0$.

We'll see much more about determinants later in the course.

1. For each linear transformation described below, determine whether or not it is invertible and, if it is invertible, find the matrix of its inverse.

(a) $P : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $P(\vec{x}) = \left(\dfrac{\vec{x} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \right) \vec{a}$, where $\vec{a}$ is some fixed nonzero vector.

(b) $Q : \mathbb{R}^2 \to \mathbb{R}^2$, defined by letting $Q(\vec{x})$ be the result of rotating $\vec{x}$ by $\frac{7\pi}{6}$ radians about the origin.

(c) $R : \mathbb{R}^2 \to \mathbb{R}^2$, defined by letting $R(\vec{x})$ be the result of reflecting $\vec{x}$ through the line $y = -x$.

(d) $S : \mathbb{R}^2 \to \mathbb{R}^3$, defined by letting $S \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$.

(e) $T : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$T(\vec{x}) = AC^{-1}B^{-1}A^{-1}BCBAC^{-1}BA^{-1}C\vec{x}$$

where A, B and C are some invertible 2×2 matrices.

2. Let A, B and C be $n \times n$ matrices. For each of the following statements, determine whether it is true or false and provide justification.

(a) If $A^k = I_n$ for some $k > 0$, then A is invertible.

(b) If A is invertible then $A^k = I_n$ for some $k > 0$.

(c) If the rank of A is n, then A is invertible.

(d) It is possible that A be invertible and the equation $A\vec{x} = \vec{b}$ have infinitely many solutions.

(e) If there are matrices B and C such that $ABC = I_n$, then B is invertible.