1. Find the kernel and image of each of the following matrices, expressing your answers as a span of as few vectors as you can.

(a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

$$\operatorname{im}(A) = \operatorname{span}\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\2\\2\end{pmatrix}, \begin{pmatrix}3\\3\\3\end{pmatrix} \right\} = \operatorname{span}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right) \xrightarrow[(\mathrm{III})-(\mathrm{I})]{(\mathrm{II})-(\mathrm{I})} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\therefore A\vec{x} = \vec{0} \Leftrightarrow \vec{x} = \begin{pmatrix} -2s-3t \\ s \\ t \end{pmatrix} = s\begin{pmatrix}-2\\1\\0\end{pmatrix} + t\begin{pmatrix}-3\\0\\1\end{pmatrix}$$

$$\Rightarrow \ker(A) = \operatorname{span}\left\{ \begin{pmatrix}-2\\1\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\1\end{pmatrix} \right\}$$

(b) $B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

$$\operatorname{rref}(B): \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{(\mathrm{I})\leftrightarrow(\mathrm{III})} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \xrightarrow[(\mathrm{III})-(\mathrm{I})]{(\mathrm{II})-(\mathrm{I})} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{(\mathrm{II})-2(\mathrm{III})} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{(\mathrm{II})\leftrightarrow(\mathrm{III})} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \xrightarrow{(\mathrm{II})-(\mathrm{III})} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\operatorname{rank}(B) = 3 \Rightarrow \operatorname{im}(B) = \mathbb{R}^3 = \operatorname{span}\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$$

$$\& \quad B\vec{x} = \vec{0} \Leftrightarrow \vec{x} = \begin{pmatrix} 0 \\ s \\ -2s \\ s \end{pmatrix} = s\begin{pmatrix}0\\1\\-2\\1\end{pmatrix}$$

$$\Rightarrow \ker(B) = \operatorname{span}\begin{pmatrix}0\\1\\-2\\1\end{pmatrix}$$

(c) $C = \begin{pmatrix} 15 & 94 & 43 & 72 & 55 & 13 \\ 0 & 81 & 46 & 46 & 21 & 54 \\ 0 & 0 & 62 & 58 & 32 & 30 \\ 0 & 0 & 0 & 49 & 29 & 11 \\ 0 & 0 & 0 & 0 & 40 & 62 \\ 0 & 0 & 0 & 0 & 0 & 12 \end{pmatrix}$

$\operatorname{rank}(C) = 6$ & C is a 6×6 matrix

$\Rightarrow C$ is invertible

$$\Rightarrow \operatorname{im}(C) = \mathbb{R}^6 = \operatorname{span}\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_6\}$$

$$\ker(C) = \{\vec{0}\}$$

2. Let $\vec{v} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ be a nonzero vector in $\mathbb{R}^3$. Show that the kernel of the linear transformation $\text{proj}_{\vec{v}} : \mathbb{R}^3 \to \mathbb{R}^3$ is the plane defined by $3x - y + z = 0$, and express this plane as the span of two vectors.

$$\text{proj}_{\vec{v}}(\vec{x}) = \left(\frac{\vec{x}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} = \vec{0} \iff \frac{\vec{x}\cdot\vec{v}}{\vec{v}\cdot\vec{v}} = 0 \iff \vec{x}\cdot\vec{v} = \vec{0}$$

$$\text{So } \ker(\text{proj}_{\vec{v}}) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 0 \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; 3x - y + 2z = 0 \right\}$$

as required.

To express this plane as a span of vectors, we find two non parallel vectors in the plane.

- Set $x = 0,\ y = 1 \rightsquigarrow z = 1$
- Set $x = 1,\ y = 0 \rightsquigarrow z = -3$

$$\text{So } \ker(\text{proj}_{\vec{v}}) = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \right\}$$