Math 290-1 Class 13

Friday 26th October 2018

Eliminating redundancy

If a list of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ is linearly dependent, then we might wish to remove the 'redundant' vectors until we have a linearly independent set. The resulting list is then a basis of $\operatorname{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r)$. (You might want to think about why this is true.)

There is a simple procedure for doing this: define

$$A = \begin{pmatrix} \vdots & \vdots & & \vdots \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_r \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

and then put A in row-echelon form (reduced or otherwise). If column i has a pivot, then keep $\vec{v}_i$; if not, then remove it from the list.

This technique is useful for finding the basis of the image of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$: first find the matrix A of T, then put A in (r)ref, and then select the columns of A (*not* of $\operatorname{rref}(A)$!) which correspond to the pivots.

Characterising linear independence

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ be a list of n-dimensional vectors, and let A be the $n \times r$ matrix whose i^{th} column is $\vec{v}_i$ for all $1 \leqslant i \leqslant r$ (just like we did above). The following statements are equivalent to saying that the vectors $\vec{v}_1, \ldots, \vec{v}_r$ are linearly independent:

- If $k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_r\vec{v}_r = \vec{0}$, then $k_1 = k_2 = \cdots = k_r = 0$;
- No $\vec{v}_i$ can be expressed as a linear combination of the other vectors $\vec{v}_j$ (for $j \neq i$);
- The only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$;
- $\ker(A) = \{\vec{0}\}$;
- $\operatorname{rank}(A) = r$.

Coordinates

Vectors $\vec{v}_1, \ldots, \vec{v}_r$ are a basis of a subspace V if they span V and are linearly independent—in this case, each vector $\vec{v}$ in V can be expressed *uniquely* as $c_1\vec{v}_1 + \cdots + c_r\vec{v}_r$ for some scalars $c_1, \ldots, c_r$. We will soon refer to the numbers $c_1, \ldots, c_r$ as the *coordinates* of $\vec{v}$ with respect to the basis $\vec{v}_1, \ldots, \vec{v}_r$.

We will focus on the exercises from Wednesday. If there's time, we might get to this one—if not, see Canvas for the worked solution!

1. Find a basis for the subspace of $\mathbb{R}^5$ consisting of all linear combinations of the following vectors:

$$\begin{pmatrix}1\\2\\1\\0\\-1\end{pmatrix}, \quad \begin{pmatrix}1\\0\\-2\\1\\1\end{pmatrix}, \quad \begin{pmatrix}7\\-2\\-7\\-2\\-1\end{pmatrix}, \quad \begin{pmatrix}-2\\2\\2\\2\\1\end{pmatrix}, \quad \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}, \quad \begin{pmatrix}7\\2\\-1\\-4\\-5\end{pmatrix}, \quad \begin{pmatrix}0\\2\\-2\\4\\3\end{pmatrix}$$