Math 290-1 Class 14

Monday 29th October 2018

Dimension

Let V be a subspace of $\mathbb{R}^n$ and let $\vec{v}_1, \dots, \vec{v}_r$ and $\vec{w}_1, \dots, \vec{w}_s$ be vectors in V.

- If $\vec{v}_1, \dots, \vec{v}_r$ are linearly independent and $\vec{w}_1, \dots, \vec{w}_s$ span V, then $r \leqslant s$;
- If $\vec{v}_1, \dots, \vec{v}_r$ and $\vec{w}_1, \dots, \vec{w}_s$ are both bases of V, then $r = s$.

That is, $\boxed{\text{any two bases of } V \text{ have the same size}}$. This quantity is called the **dimension** of V.

Some consequences:

- To find the dimension of a subspace V, all you have to do is find a basis for V and count how many vectors there are;
- Any basis of $\mathbb{R}^n$ consists of n vectors $\vec{v}_1, \dots, \vec{v}_n$;
- An $n \times n$ matrix is invertible $\Leftrightarrow$ its columns span $\mathbb{R}^n$ $\Leftrightarrow$ its columns are linearly independent;
- If $\dim(V) = r$, then vectors $\vec{v}_1, \dots, \vec{v}_r$ span V if and only if they are linearly independent.

Rank–nullity theorem

We saw on Friday that to find the basis of the image of an $m \times n$ matrix A, you put it in row-echelon form and select the columns of A corresponding to the pivots. Thus the number of vectors in a basis of im A is equal to rank of A. Thus:

$$\dim(\operatorname{im} A) = \operatorname{rank}(A)$$

Likewise, to find the basis of the kernel of a matrix A, you solve $A\vec{x} = \vec{0}$ by Gauss–Jordan elimination and factor out the parameters. That is, the number of vectors in a basis of $\ker A$ is equal to the number of free variables in the system $A\vec{x} = \vec{0}$. Thus:

$$\dim(\ker A) = n - \operatorname{rank}(A)$$

Putting these facts together yields the **rank–nullity theorem**:

$$\underbrace{\dim(\operatorname{im} A)}_{\text{rank}} + \underbrace{\dim(\ker A)}_{\text{nullity}} = \underbrace{n}_{\text{dim. of domain}} \qquad (= \#\text{ columns of } A)$$

1. Find the dimensions of the following subspaces.

(a) The plane in $\mathbb{R}^3$ defined by $2x - y + z = 0$.

(b) The image of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ whose matrix is $\begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 0 \\ 0 & -1 & 1 \\ -1 & 1 & -3 \end{pmatrix}$.

(c) The kernel of the linear transformation T from part (b).

(d) The set of solutions $\vec{x}$ in $\mathbb{R}^n$ to the equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$.

2. For each of the following statements, fill in the sign ($<$, $\leqslant$, $=$, $\geqslant$, $>$) or write **?** if none of these apply.

(a) If U is a subspace of $\mathbb{R}^n$, then $\dim(U)$ ☐ n.

(b) If A and B are matrices, then $\text{rank}(AB)$ ☐ $\text{rank}(A)$.

(c) If A is an $n \times n$ matrix, then $\text{rank}(A + I_n)$ ☐ $\text{rank}(A)$.

(d) If $m \leqslant n$ and V is the subspace of $\mathbb{R}^n$ consisting of all solutions $\vec{x}$ to the linear system below, then $\dim(V)$ ☐ $n - m$.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \quad\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$