1. For each of the following specifications of a subspace V of $\mathbb{R}^n$, basis $\mathfrak{B}$ of V and vector $\vec{a}$ in V, find the coordinate vector $[\vec{a}]_{\mathfrak{B}}$.

(a) $V = \mathbb{R}^3$, $\mathfrak{B} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{a} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ -1 & 0 & 1 & 3 \\ 0 & 2 & 1 & 3 \end{array}\right) \xrightarrow{(II)+(I)} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 1 & 3 \end{array}\right)$$

$$\xrightarrow[(III)-(II)]{(I)-(II)} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow[(II)\div 2]{(I)+(III)} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \begin{cases} x_1 = -2 \\ x_2 = 1 \\ x_3 = 1 \end{cases}$$

$$\Rightarrow \left[\begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}\right]_{\mathfrak{B}} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

(b) $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbb{R}^3 : 3x + y = 0 \right\}$, $\mathfrak{B} = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$, $\vec{a} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$.

$$\left(\begin{array}{cc|c} 1 & -1 & -1 \\ -3 & 3 & 3 \\ 0 & 1 & 3 \end{array}\right) \xrightarrow{(II)+3(I)} \left(\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{(I)+(III)} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{array}\right) \begin{cases} x_1 = 2 \\ 0 = 0 \\ x_2 = 3 \end{cases}$$

$$\Rightarrow \left[\begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}\right]_{\mathfrak{B}} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Parts (c) and (d) are on the next page…

(c) $V=\mathbb{R}^3$, $\mathfrak{B}=\vec{e}_1,\vec{e}_2,\vec{e}_3$, $\vec{a}=\begin{pmatrix}1\\2\\3\end{pmatrix}$.

$$\begin{pmatrix}1\\2\\3\end{pmatrix} = \vec{e}_1 + 2\vec{e}_2 + 3\vec{e}_3$$

$$\Rightarrow \left[\begin{pmatrix}1\\2\\3\end{pmatrix}\right]_{\mathfrak{B}} = \begin{pmatrix}1\\2\\3\end{pmatrix}$$

(d) $V=\mathbb{R}^3$, $\mathfrak{B}=\vec{e}_3,\vec{e}_1,\vec{e}_2$, $\vec{a}=\begin{pmatrix}1\\2\\3\end{pmatrix}$.

$$\begin{pmatrix}1\\2\\3\end{pmatrix} = 3\vec{e}_3 + \vec{e}_1 + 2\vec{e}_2$$

$$\Rightarrow \left[\begin{pmatrix}1\\2\\3\end{pmatrix}\right]_{\mathfrak{B}} = \begin{pmatrix}3\\1\\2\end{pmatrix}$$

2. Find $\vec{a}$ given that $[\vec{a}]_{\mathfrak{B}}=\begin{pmatrix}1\\1\\-2\\3\end{pmatrix}$, where $\mathfrak{B}=\begin{pmatrix}4\\0\\2\\1\\1\end{pmatrix},\begin{pmatrix}1\\2\\0\\-7\\2\end{pmatrix},\begin{pmatrix}4\\1\\-3\\1\\0\end{pmatrix},\begin{pmatrix}-2\\-1\\1\\3\\0\end{pmatrix}$.

$$\vec{a} = \begin{pmatrix}4\\0\\2\\1\\1\end{pmatrix} + \begin{pmatrix}1\\2\\0\\-7\\2\end{pmatrix} - 2\begin{pmatrix}4\\1\\-3\\1\\0\end{pmatrix} + 3\begin{pmatrix}-3\\-1\\1\\3\\6\end{pmatrix}$$

$$= \begin{pmatrix}4+1-8+9\\0+2-2-3\\2+0+6+3\\1-7-2+9\\1+2-0+0\end{pmatrix} = \begin{pmatrix}6\\-3\\11\\1\\3\end{pmatrix}$$