Math 290-1 Class 16

Friday 2nd November 2018

∼ Class next week will meet in University Hall 122 ∼

Change of basis

Recall that a list of vectors $\mathfrak{B} = \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is a basis of $\mathbb{R}^n$ if and only if the matrix S is invertible, where S is the matrix whose columns are the vectors in $\mathfrak{B}$:

$$S = \begin{pmatrix} \vdots & \vdots & & \vdots \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_r \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

The matrix S is the **transition matrix** of $\mathfrak{B}$. Notice that

$$S \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

and hence $S[\vec{a}]_{\mathfrak{B}} = \vec{a}$ for all vectors $\vec{a}$.

We can think of S as a matrix that **decodes** $\mathfrak{B}$-coordinates: given a coordinate vector $[\vec{a}]_{\mathfrak{B}}$, applying S gives you back the vector $\vec{a}$. Conversely, S^{-1} **encodes** $\mathfrak{B}$-coordinates: given a vector $\vec{a}$, applying S^{-1} gives you its $\mathfrak{B}$-coordinate vector $[\vec{a}]_{\mathfrak{B}}$.

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ and a basis $\mathfrak{B}$ of $\mathbb{R}^n$, the **$\mathfrak{B}$-matrix** of T is the $n \times n$ matrix B satisfying $B[\vec{x}]_{\mathfrak{B}} = [T(\vec{x})]_{\mathfrak{B}}$. Note that we must have

$$B\vec{e}_i = B[\vec{v}_i]_{\mathfrak{B}} = [T(\vec{v}_i)]_{\mathfrak{B}}$$

and so the i^{th} column of B is given by $[T(\vec{v}_i)]_{\mathfrak{B}}$. Intuitively, B is a matrix in a dream world where the vectors in $\mathfrak{B}$ are treated like the standard basis vectors—the i^{th} column of B is obtained by pretending that the vector $\vec{v}_i$ is the i^{th} standard basis vector, applying T and then reading off the $\mathfrak{B}$-coordinates of the result.

If B is the $\mathfrak{B}$-matrix of T, its standard matrix A is given by $A = SBS^{-1}$:

$$\vec{x} \xmapsto[\mathfrak{B}\text{-coordinates}]{\text{encode as}} S^{-1}\vec{x} \xmapsto[\mathfrak{B}\text{-matrix}]{\text{apply}} BS^{-1}\vec{x} \xmapsto[\mathfrak{B}\text{-coordinates}]{\text{decode from}} SBS^{-1}\vec{x}$$

If A is the standard matrix of T, its $\mathfrak{B}$-matrix B is given by $B = S^{-1}AS$:

$$\vec{x} \xmapsto[\mathfrak{B}\text{-coordinates}]{\text{decode from}} S\vec{x} \xmapsto[\text{standard matrix}]{\text{apply}} AS\vec{x} \xmapsto[\mathfrak{B}\text{-coordinates}]{\text{encode as}} S^{-1}AS\vec{x}$$

This relationship between A and B is called **similarity**.

1. Find the matrix of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ which satisfies

$$T\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \quad T\begin{pmatrix} -1 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \\ 2 \end{pmatrix}$$

[*Hint:* First find the $\mathfrak{B}$-matrix of T where $\mathfrak{B}$ is a cleverly chosen basis.]

2. For each of the following statements, determine whether it is true or false.

(a) If the $\mathfrak{B}$-matrix of A is invertible, then A is invertible.

(b) If A and B are similar $n \times n$ matrices, then there are bases $\mathfrak{B}$ and $\mathfrak{C}$ of $\mathbb{R}^n$ such that B is the $\mathfrak{B}$-matrix of A and A is the $\mathfrak{C}$-matrix of B.

(c) If matrices A and B commute, then A and B are similar.

(d) For all vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$, all scalars a and b, and all bases $\mathfrak{B}$, we have

$$[a\vec{x} + b\vec{y}]_{\mathfrak{B}} = a[\vec{x}]_{\mathfrak{B}} + b[\vec{y}]_{\mathfrak{B}}$$

3. Let $\mathfrak{B} = \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be a basis of $\mathbb{R}^n$, let $\mathfrak{C} = \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$ be a basis of $\mathbb{R}^m$, and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. In terms of the (standard) matrix A of T, find a matrix Q such that

$$Q[\vec{x}]_{\mathfrak{B}} = [T(\vec{x})]_{\mathfrak{C}}$$