Math 290-1 Class $17\frac{1}{2}$

Supplementary notes

$\mathfrak{B}$-matrices encore une fois

Let $\mathfrak{B} = \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ be a basis of $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation.

The $\mathfrak{B}$-matrix of T is the $n \times n$ matrix B which describes the behaviour of T with the world viewed in terms of $\mathfrak{B}$-coordinates. Notice that $\vec{e}_i = [\vec{v}_i]_{\mathfrak{B}}$, and so the i^{th} column of B (which is $B\vec{e}_i$) should tell us the $\mathfrak{B}$-coordinates of $T(\vec{v}_i)$. That is,

$$i^{\text{th}} \text{ column of } B = B\vec{e}_i = B[\vec{v}_i]_{\mathfrak{B}} = [T(\vec{v}_i)]_{\mathfrak{B}} = \mathfrak{B}\text{-coordinates of } T(\vec{v}_i)$$

This means that you can find the $\mathfrak{B}$-matrix B by first finding $T(\vec{v}_i)$, and then putting the result into $\mathfrak{B}$-coordinates. This method is useful when $T(\vec{v}_i)$ has an obvious expression in terms of the vectors in $\mathfrak{B}$. [The other method is to find the standard matrix A of T and then compute $B = S^{-1}AS$, where S is the transition matrix of $\mathfrak{B}$. This method is failsafe but very computationally cumbersome.]

Diagonal $\mathfrak{B}$-matrices

An $n \times n$ matrix B is diagonal if all its off-diagonal entries are zero. That is, there are scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ such that

$$B = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

If $\mathfrak{B} = \vec{v}_1, \dots, \vec{v}_n$ is a basis of $\mathbb{R}^n$ and B is the $\mathfrak{B}$-matrix of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, this says that $T(\vec{v}_i) = \lambda_i \vec{v}_i$ for each basis vector $\vec{v}_i$.

That is, the $\mathfrak{B}$-matrix of T is diagonal if and only if $T(\vec{v}_i)$ is parallel to $\vec{v}_i$ for each basis vector $\vec{v}_i$.

An interesting question to ask is: given a linear transformation T, when does there exist a basis $\mathfrak{B}$ with respect to which the matrix of T is diagonal?

Such a basis must consist of (nonzero!) vectors $\vec{v}$ such that $T(\vec{v}) = \lambda\vec{v}$ for some scalar λ. So let's explore the consequences of this further. First let A be the standard matrix of T. Then

$$\begin{aligned}
& T(\vec{v}) = \lambda\vec{v} \text{ for some nonzero } \vec{v} & & \\
\Leftrightarrow\ & A\vec{v} = \lambda\vec{v} \text{ for some nonzero } \vec{v} & & \text{since } A \text{ is the matrix of } T \\
\Leftrightarrow\ & (A - \lambda I)\vec{v} \text{ for some nonzero } \vec{v} = \vec{0} & & \text{rearranging} \\
\Leftrightarrow\ & \vec{v} \in \ker(A - \lambda I) \text{ for some nonzero } \vec{v} & & \text{by definition of kernel} \\
\Leftrightarrow\ & A - \lambda I \text{ is not invertible} & & \text{(fact from the course)} \\
\Leftrightarrow\ & \det(A - \lambda I) = 0 & & \text{(fact from the course)}
\end{aligned}$$

Woah! We just showed that $T(\vec{v}) = \lambda\vec{v}$ for some nonzero vector $\vec{v}$ if and only if $\det(A - \lambda I) = 0$.

Thus, the only way T can have a diagonal $\mathfrak{B}$-matrix is if its diagonal entries $\lambda = \lambda_i$ are solutions to the equation $\det(A - \lambda I) = 0$. Given such a value λ_i, the vector $\vec{v}_i$ can then be taken to be anything in the kernel of $A - \lambda_i I$.

Example.
Find a basis $\mathfrak{B}$ with respect to which the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\vec{x}) = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

is diagonal.

Solution. As we discussed above, if we're to have $T(\vec{v}) = \lambda\vec{v}$ for some scalar λ and nonzero vector $\vec{v}$, then the matrix $A - \lambda I_2$ must have determinant zero, since $\vec{v} \in \ker(A - \lambda I_2)$. Now

$$A - \lambda I_2 = \begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix} \Rightarrow \det(A - \lambda I_2) = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5$$

Now $\lambda^2 - 4\lambda - 5 = (\lambda+1)(\lambda-5)$, so $\det(A - \lambda I_2) = 0$ if and only if $\lambda = -1$ or $\lambda = 5$.

- When $\lambda = -1$, we have $A - \lambda I_2 = \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}$. The vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is in the kernel of this matrix.
- When $\lambda = 5$, we have $A - \lambda I_2 = \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix}$. The vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is in the kernel of this matrix.

So let $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and let $\mathfrak{B} = \vec{v}_1, \vec{v}_2$.

Now let's check our work. First note that $\mathfrak{B}$ is a basis of $\mathbb{R}^2$ since $\vec{v}_1$ and $\vec{v}_2$ are linearly independent. Moreover:

$$T(\vec{v}_1) = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\vec{v}_1 \quad \Rightarrow \quad [T(\vec{v}_1)]_{\mathfrak{B}} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$T(\vec{v}_2) = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 5\vec{v}_2 \quad \Rightarrow \quad [T(\vec{v}_2)]_{\mathfrak{B}} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

Hence the $\mathfrak{B}$-matrix of T is $\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$, which is diagonal, as claimed.

Eigenwhat?

After the next midterm, we will explore this in a lot more detail. A vector $\vec{v}$ such that $T(\vec{v}) = \lambda\vec{v}$ for some scalar λ is called an **eigenvector** of T, and λ is its **eigenvalue**. A basis of $\mathbb{R}^n$ consisting of eigenvectors of T is called an **eigenbasis**, and a general fact is that the matrix of a linear transformation T with respect to an eigenbasis is diagonal.