1. Find the determinants of the following matrices.

(a) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \det(d) - b \det(c) = \underline{\underline{ad - bc}}$$

Expansion along 1^{st} row

Same answer!

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = -b \det(c) + d \det(a) = \underline{\underline{ad - bc}}$$

Expansion along 2^{nd} column

(b) $\begin{pmatrix} 3 & -5 & 8 \\ 0 & 0 & -11 \\ 2 & -1 & 3 \end{pmatrix}$

Hard way: expand along 1^{st} row:

$$3 \begin{vmatrix} 0 & -11 \\ -1 & 3 \end{vmatrix} - (-5) \begin{vmatrix} 0 & -11 \\ 2 & 3 \end{vmatrix} + 8 \begin{vmatrix} 0 & 0 \\ 2 & -1 \end{vmatrix}$$

$$= 3(0 - 11) + 5(0 + 22) + 8(0 - 0)$$

$$= -33 + 110$$

$$= \underline{\underline{77}}$$

Easy way: expand along 2^{nd} row:

$$-0 \begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} + 0 \begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} - (-11) \begin{vmatrix} 3 & -5 \\ 2 & -1 \end{vmatrix}$$

$$= 11(-3 + 10)$$

$$= \underline{\underline{77}}$$

Same answer!

(c) $\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$ Expand along row 1 :

$$\begin{vmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{vmatrix} = \underbrace{\begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix}}_{A} + \underbrace{\begin{vmatrix} -1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix}}_{B} - \underbrace{\begin{vmatrix} -1 & 1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 1 \end{vmatrix}}_{C} + \underbrace{\begin{vmatrix} -1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & -1 \end{vmatrix}}_{D}$$

Notice that $B_{11} = C_{21} = D_{31}$, $B_{12} = C_{22} = D_{32}$, $B_{13} = C_{23} = D_{33}$

So expanding along the "$-1\ -1\ -1$" rows of B, C, D gives

that $\det(B) = -\det(C) = \det(D)$

$\uparrow$ $\because$ 2nd row is $- + -$ but 1st, 3rd are $+ - +$.

$$\det(A) = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ -1 & 1 \end{vmatrix} - \begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix} = 0 + (-2) - 2 = -4$$

$$\det(B) = -\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ -1 & 1 \end{vmatrix} - \begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix} = -0 + (-2) - 2 = -4$$

$$\Rightarrow \begin{vmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{vmatrix} = (-4) + (-4) - 4 + (-4) = \underline{\underline{-16}}$$

(d) $\begin{pmatrix} 18 & 13 & 30 & 87 & 89 \\ 62 & 88 & 19 & 43 & 22 \\ 96 & 13 & 14 & 44 & 65 \\ 18 & 13 & 30 & 87 & 89 \\ 44 & 46 & 26 & 19 & 87 \end{pmatrix}$

$$\text{Row}_1 = \text{Row}_4 \Rightarrow \text{rref}(A) = \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \\ 0\ 0\ 0\ 0\ 0 \end{pmatrix}$$

$\Rightarrow \underline{\underline{\det(A) = 0}}$ $\therefore$ A is not invertible

2. Find the volume of the parallelepiped determined by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, where

$$\vec{a} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \qquad \vec{c} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

The unit ~~square~~ cube is the parallelepiped determined by the standard basis vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ — it has volume 1.

Let $A = \begin{pmatrix} \vdots & \vdots & \vdots \\ \vec{a} & \vec{b} & \vec{c} \\ \vdots & \vdots & \vdots \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 3 & 0 \\ -2 & 4 & 1 \end{pmatrix}$. Then A maps the unit cube onto the parallelepiped in question, since

$A\vec{e}_1 = \vec{a}$, $A\vec{e}_2 = \vec{b}$, $A\vec{e}_3 = \vec{c}$.

$\Rightarrow$ volume $= |\det(A)|$.

Expand down 3^{rd} column: $(-1)\begin{vmatrix} 1 & 3 \\ -2 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = -(4+6)+(3-2)$
$= -9$

$\Rightarrow$ volume $= |-9| = \underline{\underline{9}}$

3. For which values of k is the following matrix non-invertible?

$$\begin{pmatrix} 1 & 2 & -1 \\ k & 0 & 1 \\ 0 & k & 1 \end{pmatrix}$$

$$\det \begin{pmatrix} 1 & 2 & -1 \\ k & 0 & 1 \\ 0 & k & 1 \end{pmatrix} \underset{\substack{\uparrow \\ \text{exp. along} \\ 3^{rd} \text{ row}}}{=} -k \begin{vmatrix} 1 & -1 \\ k & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ k & 0 \end{vmatrix} = -k(1+k) + (0-2k)$$

$$= -k - k^2 - 2k$$

$$= -k(k+3)$$

So $\det(A) = 0 \iff -k(k+3) = 0$

$\Updownarrow \qquad \iff \underline{\underline{k=0}}$ or $\underline{\underline{k=3}}$

A is non-invertible.

4. Explain why the determinant of an upper-triangular matrix is the product of its diagonal entries:

$$\det \underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}}_{A} = a_{11} \times a_{22} \times \cdots \times a_{nn}$$

Expanding down the 1st column gives

$$\det(A) = a_{11} \cdot \det \begin{pmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

$$+ 0 \times (\text{a bunch of stuff})$$

But $\boxed{\text{this}}$ is another upper-triangular matrix! So just repeat the argument. We continue pulling out factors of the form a_{kk} until we're left with

$$\det(A) = a_{11} a_{22} \cdots a_{nn}$$

as required.