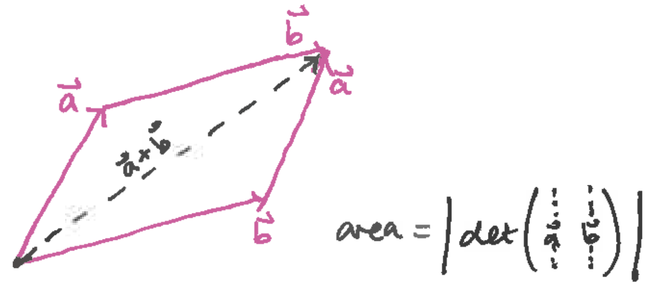
Math 290-1 Class 21

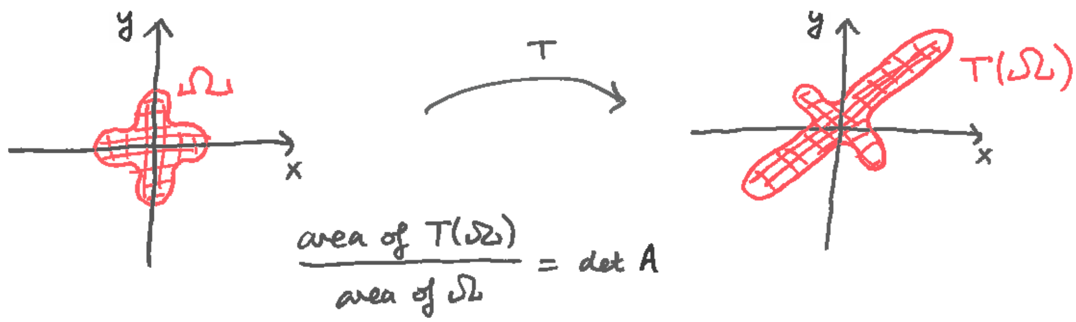
Wednesday 14th November 2018

Determinants in two dimensions

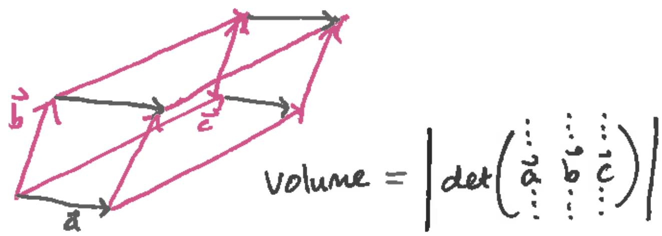


Any two vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^2$ determine a (possibly flat) parallelogram in $\mathbb{R}^2$. Its area is $|\det(A)|$, where A is the 2×2 matrix whose columns are $\vec{a}$ and $\vec{b}$.

More generally, if Ω is a region of $\mathbb{R}^2$ with area $k \geqslant 0$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation represented by a matrix A, then $T(\Omega) = \{T(\vec{x}) : \vec{x} \text{ is in } \Omega\}$ is a region of $\mathbb{R}^2$ with area $k \cdot |\det(A)|$.

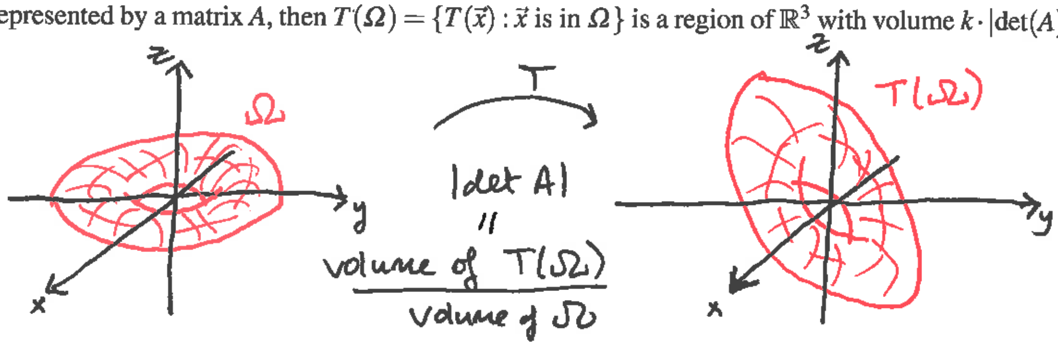


Determinants in three dimensions



Any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ in $\mathbb{R}^3$ determine a (possibly flat) parallelepiped in $\mathbb{R}^3$. Its area is $|\det(A)|$, where A is the 3×3 matrix whose columns are $\vec{a}$, $\vec{b}$ and $\vec{c}$.

More generally, if Ω is a region of $\mathbb{R}^3$ with volume $k \geqslant 0$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation represented by a matrix A, then $T(\Omega) = \{T(\vec{x}) : \vec{x} \text{ is in } \Omega\}$ is a region of $\mathbb{R}^3$ with volume $k \cdot |\det(A)|$.



In light of the above facts, we will refer to the quantity $|\det(A)|$ as the *expansion factor* of A.

1. Let Ω be the triangle in $\mathbb{R}^2$ whose vertices have position vectors $\begin{pmatrix}-1\\1\end{pmatrix}$, $\begin{pmatrix}2\\3\end{pmatrix}$ and $\begin{pmatrix}4\\0\end{pmatrix}$.

Find the area of Ω.

2. Let T be the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(\vec{x}) = \begin{pmatrix} 6 & -1 & 1 \\ -3 & 1 & 2 \\ 4 & -1 & -3 \end{pmatrix} \vec{x}$$

Given that a region Ω of $\mathbb{R}^3$ is transformed via T to a cylinder of height 3 and radius 2, find the volume of Ω.

3. Find the area of a regular n-gon inscribed in a circle of radius 1.

4. For each of the following statements, determine whether it is always, sometimes or never true.

(a) Let A be a 3×3 matrix and let $\Omega \subseteq \mathbb{R}^3$ be a two-dimensional region with area $k \geqslant 0$. Then the area of $T(\Omega)$ is $k \cdot |\det(A)|$.

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a rotation or reflection and let Ω be a region in $\mathbb{R}^3$. Then $T(\Omega)$ and Ω have the same volume.

(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation and let $\vec{v}$ and $\vec{w}$ be vectors such that $T(\vec{v}) = \lambda \vec{v}$ and $T(\vec{w}) = \mu w$ for some scalars $\lambda \neq \mu$. Then the expansion factor of T is $|\lambda \mu|$.