Math 290-1 Class 24

Monday 26th November 2018

Diagonal matrices

A matrix D is **diagonal** if $d_{ij} = 0$ for all $i \neq j$—that is, only its diagonal entries d_{ii} may be nonzero. A matrix is diagonal if and only if the standard basis vectors are all eigenvectors. In $\mathbb{R}^2$:

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Diagonal matrices are very well-behaved. For example, the matrix D^n can be computed by simply raising each diagonal entry of D to its n^{th} power; the eigenvalues of D are its diagonal entries; the determinant of D is the product of its diagonal entries; the (geometric and algebraic) multiplicity of each eigenvalue is equal to the number of times it appears on the diagonal; and many more fun facts (try to come up with some by yourself).

A matrix A is **diagonalisable** if it is similar to a diagonal matrix, i.e. if there is a diagonal matrix D and an invertible matrix S such that $D = S^{-1}AS$ (or equivalently $A = SDS^{-1}$). To *diagonalise* a matrix A is to find a diagonal matrix D similar to A.

The following conditions on a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ with $n \times n$ matrix A are equivalent:

- A is diagonalisable;
- There is a basis $\mathfrak{B}$ of $\mathbb{R}^n$ with respect to which the matrix of T is diagonal;
- There is a basis $\mathfrak{B}$ of $\mathbb{R}^n$ consisting of eigenvectors of T (called an **eigenbasis** for T);
- The sum of the geometric multiplicities of the eigenvalues of T is equal to n;
- The characteristic polynomial $f_A(\lambda)$ can be fully factorised, and the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity.

We can piece together this information to inform strategies for either diagonalising a diagonalisable matrix, or showing that a matrix is not diagonalisable.

Step 1 Solve $f_A(\lambda) = 0$ to find the eigenvalues of A and their algebraic multiplicities;

Step 2 Find a basis for each eigenspace $E_\lambda = \ker(A - \lambda I_n)$ of A;

Step 3 If some eigenvalue λ has geometric multiplicity $<$ algebraic multiplicity, then conclude that A is not diagonalisable...

Step 4 ...otherwise, piece together the bases of the eigenspaces of A to form a basis $\mathfrak{B}$ of $\mathbb{R}^n$—then the $\mathfrak{B}$-matrix $D = S^{-1}AS$ of A is diagonal (where S is the transition matrix of $\mathfrak{B}$).

1. Diagonalise the following matrix.

$$A = \begin{pmatrix} -2 & 3 & 3 \\ 0 & -5 & -3 \\ 0 & 6 & 4 \end{pmatrix}$$

2. Compute $\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}^{1000}$.

3. For each of the following statements, determine whether it is always, sometimes or never true.

(a) An $n \times n$ matrix A with n distinct eigenvalues is diagonalisable.

(b) Let A be a 3×3 diagonalisable matrix. Then $f_A(\lambda) = \lambda^3 + \lambda^2 + \lambda + 1$.

(c) Let A be a non-diagonalisable $n \times n$ matrix. Then A^2 is not diagonalisable.