Math 290-1 Class 25

Wednesday 28th November 2018

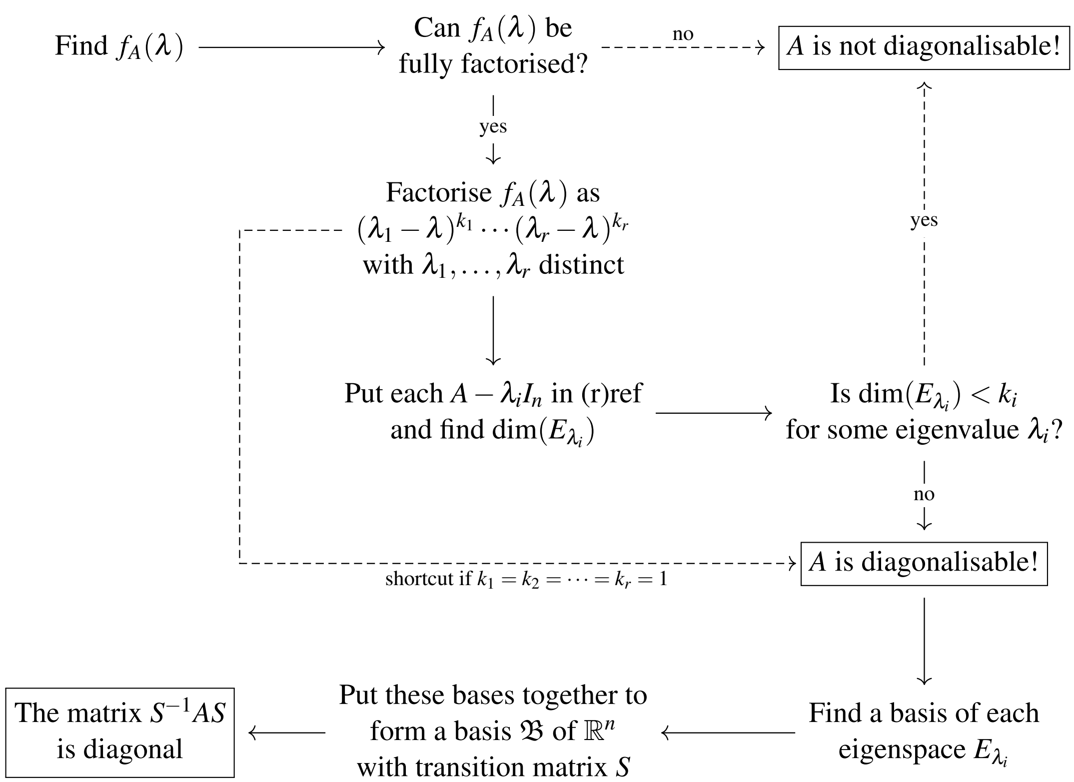
Criteria for diagonalisability

Recall that a matrix A is **diagonalisable** if it is similar to a diagonal matrix, i.e. if there is a diagonal matrix D and an invertible matrix S such that $D = S^{-1}AS$ (or equivalently $A = SDS^{-1}$).

The following conditions on a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ with $n \times n$ matrix A are equivalent:

- A is diagonalisable;
- There is a basis $\mathfrak{B}$ of $\mathbb{R}^n$ with respect to which the matrix of T is diagonal;
- There is a basis $\mathfrak{B}$ of $\mathbb{R}^n$ consisting of eigenvectors of T (called an **eigenbasis** for T);
- The sum of the geometric multiplicities of the eigenvalues of T is equal to n;
- The characteristic polynomial $f_A(\lambda)$ can be fully factorised, and the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity.

This provides us with a useful algorithm for determining if a matrix A is diagonalisable, and if it is, finding a basis with respect to which the matrix of $T(\vec{x}) = A\vec{x}$ is diagonal.



1. For each of the following matrices, determine whether or not it is diagonalisable. If it is, write down a diagonal matrix that it is similar to—you do not need to find an eigenbasis.

(a) $A = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$

(b) $B = \begin{pmatrix} -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

(c) $C = \begin{pmatrix} -1 & 4 & 2 \\ 1 & 2 & 2 \\ -2 & -2 & -3 \end{pmatrix}$ — you may assume that $f_C(\lambda) = -\lambda^3 - 2\lambda^2 + \lambda + 2$.

2. [*Repeated from Monday*] For each of the following statements, determine whether it is always, sometimes or never true.

(a) An $n \times n$ matrix A with n distinct eigenvalues is diagonalisable.

(b) Let A be a 3×3 diagonalisable matrix. Then $f_A(\lambda) = \lambda^3 + \lambda^2 + \lambda + 1$.

(c) Let A be a non-diagonalisable $n \times n$ matrix. Then A^2 is not diagonalisable.

3. [*Bretscher §7.1 Q72, modified*] Consider the growth of a lilac bush. At the beginning of its life the bush has one branch, and during each subsequent year of its life, each branch that already existed at the beginning of the previous year grows two new branches. (We assume that no branches ever die.)

Let $a(t)$ be the number of branches that the bush already had at the beginning of year t, and let $n(t)$ be the number of new branches that the bush grows during year t, where 'year t' is the year that *ends* when the bush is t years old.

Find closed-form expressions for $a(t)$ and $n(t)$.