1. For each of the following matrices, determine whether or not it is diagonalisable. If it is, write down a diagonal matrix that it is similar to—you do not need to find an eigenbasis.

(a) $\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$

$tr(A) = 5, \quad det(A) = 7$

$\Rightarrow f_A(\lambda) = \lambda^2 - 5\lambda + 7$

discriminant $= (-5)^2 - 4 \cdot 7 = 25 - 28 = -3 < 0$

$\Rightarrow f_A(\lambda)$ has no real roots

$\Rightarrow \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$ is <u>not diagonalisable</u> (over $\mathbb{R}$)

(b) $\begin{pmatrix} -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

$det(B - \lambda I_4) = (-1-\lambda)^2 \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (-1-\lambda)^2(\lambda^2 - 4\lambda + 3)$

So alg. mult. of -1 is ≥ 2. But

$B - (-1)I_4 = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$rank(B + I_4) = 3 \Rightarrow dim(E_{-1}) = 4 - 3 = 1 < 2$

So the geom. mult. of -1 is less than its alg. mult. $\leadsto$ <u>not diagonalisable</u>

(c) $\begin{pmatrix} -1 & 4 & 2 \\ 1 & 2 & 2 \\ -2 & -2 & -3 \end{pmatrix}$

$f_C(\lambda) = \underbrace{-\lambda^3 - 2\lambda^2}_{(= (-\lambda^2)(\lambda+2))} + \lambda + 2$

$= (\lambda + 2)(-\lambda^2 + 1)$

$= (\lambda + 2)(1 - \lambda)(1 + \lambda)$

The eigenvalues of C are $-2, 1, -1$

which are distinct

$\Rightarrow$ <u>C is diagonalisable</u> & C is similar to $\begin{pmatrix} -2 & & \\ & 1 & \\ & & -1 \end{pmatrix}$

2. [*Repeated from Monday*] For each of the following statements, determine whether it is always, sometimes or never true.

(a) An $n \times n$ matrix A with n distinct eigenvalues is diagonalisable.

Always! Each eigenvalue λ_i has geometric multiplicity ≥ 1 & eigenvectors with distinct eigenvalues are LI so letting $\vec{v}_i$ be an eigenvector w/ eigenvalue λ_i we have $\vec{v}_1, \ldots, \vec{v}_n$ are LI

$\Rightarrow$ they form a basis of $\mathbb{R}^n$

$\Rightarrow$ A is diagonalisable.

(b) Let A be a 3×3 diagonalisable matrix. Then $f_A(\lambda) = \lambda^3 + \lambda^2 + \lambda + 1$.

Never! $\lambda^3 + \lambda^2 + \lambda + 1$

$= \lambda^2(\lambda + 1) + \lambda + 1$

$= (\lambda + 1)\underbrace{(\lambda^2 + 1)}$ ← does not factorise over $\mathbb{R}$

$\Rightarrow$ A is not diagonalisable over $\mathbb{R}$.

(c) Let A be a non-diagonalisable $n \times n$ matrix. Then A^2 is not diagonalisable.

Sometimes!

True → Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

→ A is not diagonalisable: $f_A(\lambda) = (1-\lambda)^2$

but $A - I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \dim(E_1) = 1 < 2$

→ $A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is not diagonalisable: $f_{A^2}(\lambda) = (1-\lambda)^2$

but $A^2 - I = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \Rightarrow \dim(E_1) = 1 < 2$.

False → Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

→ A is not diagonalisable: $f_A(\lambda) = \lambda^2$

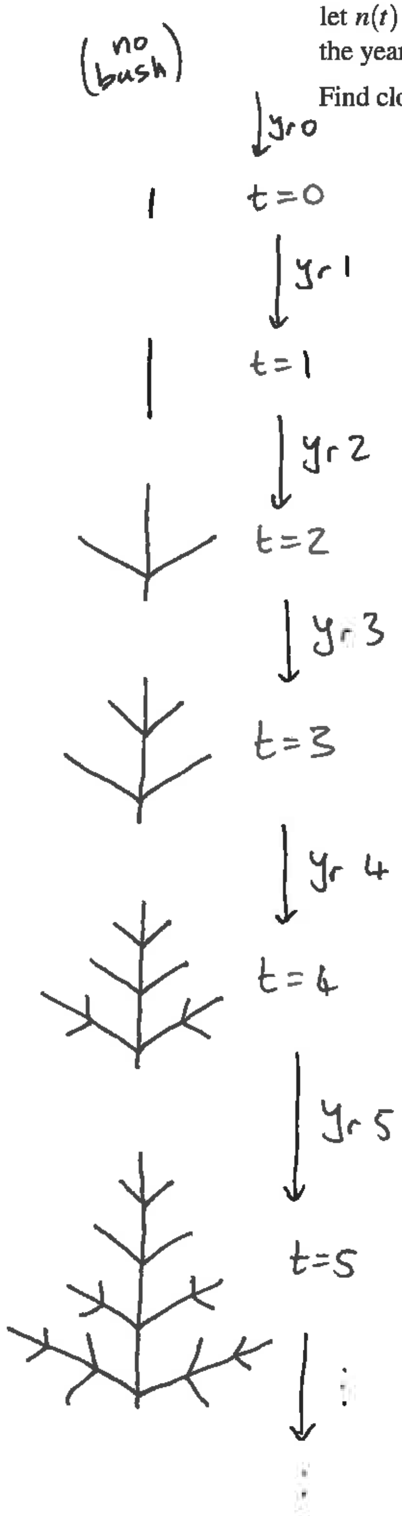
but $A = A - 0I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \dim(E_0) = 1 < 2$

→ $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ which is diagonal so is definitely diagonalisable.

3. [*Bretscher §7.1 Q72, modified*] Consider the growth of a lilac bush. At the beginning of its life the bush has one branch, and during each subsequent year of its life, each branch that already existed at the beginning of the previous year grows two new branches. (We assume that no branches ever die.)

Let $a(t)$ be the number of branches that the bush already had at the beginning of year t, and let $n(t)$ be the number of new branches that the bush grows during year t, where 'year t' is the year that *ends* when the bush is t years old.

Find closed-form expressions for $a(t)$ and $n(t)$.



$\underline{n(0) = 1}$ and $\underline{a(0) = 0}$

The description tells us that

$$\begin{cases} a(t+1) = a(t) + n(t) & \because \text{ branches don't die} \\ n(t+1) = 2a(t) & \because \text{ old branches grow 2 new branches each year} \end{cases}$$

$$\Rightarrow \begin{pmatrix} a(t+1) \\ n(t+1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} a(t) \\ n(t) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a(t) \\ n(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}^t \begin{pmatrix} a(0) \\ n(0) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}^t \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$. Then $f_A(\lambda) = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1)$

$$A - 2I = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \Rightarrow E_2 = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$

$$A + I = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \Rightarrow E_{-1} = \operatorname{span}\left\{\begin{pmatrix} -1 \\ 2 \end{pmatrix}\right\}$$

$$\Rightarrow A = S\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}S^{-1} \text{ where } \underbrace{S = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}}$$

$$\hookrightarrow S^{-1} = \frac{1}{2-(-1)}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\Rightarrow A^t = S\begin{pmatrix} 2^t & 0 \\ 0 & (-1)^t \end{pmatrix}S^{-1}$$

$$= \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2^t & 0 \\ 0 & (-1)^t \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2^{t+1} & 2^t \\ (-1)^{t+1} & (-1)^t \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2^{t+1} + (-1)^t & 2^t + (-1)^{t+1} \\ 2^{t+1} + 2(-1)^{t+1} & 2^t + 2(-1)^t \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a(t) \\ n(t) \end{pmatrix} = A^t\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2^t + (-1)^{t+1} \\ 2^t + 2(-1)^t \end{pmatrix}$$