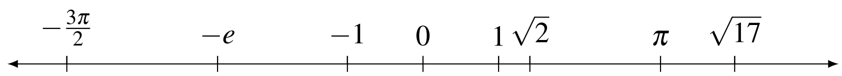
Math 290-1 Class 26

Friday 30th November 2018

Complex numbers

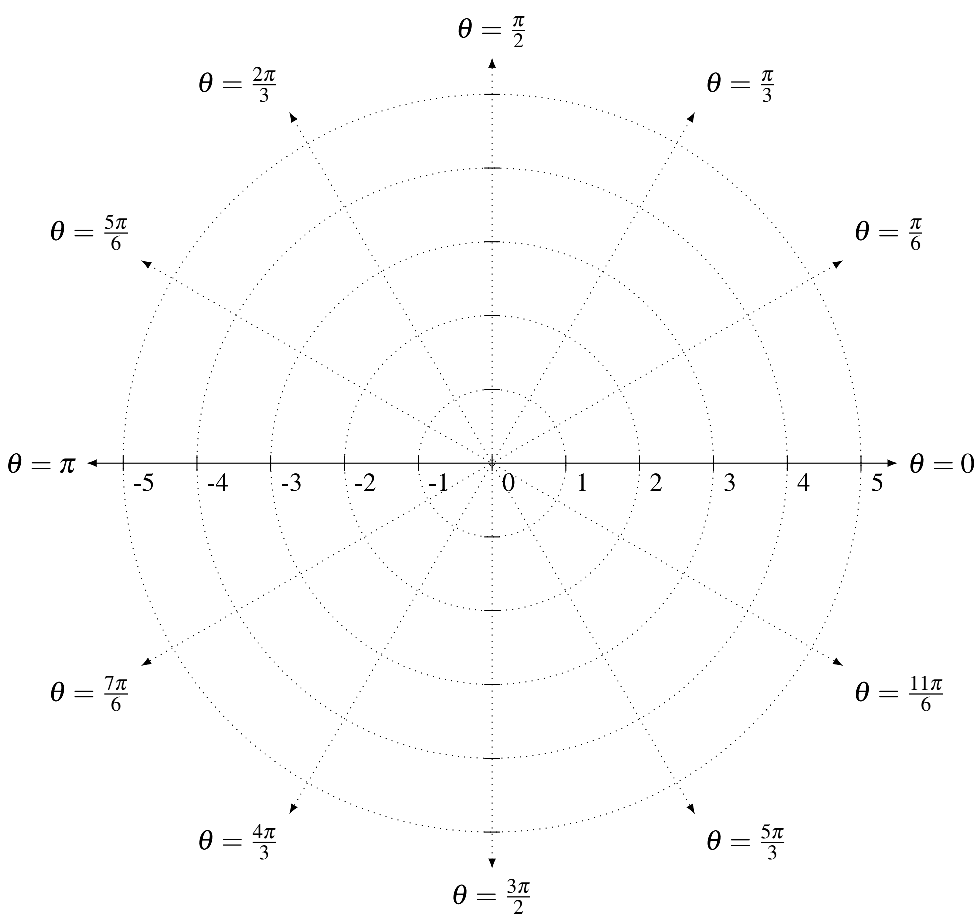
We visualise real numbers as points on a *number line*:



The arithmetic operations can be interpreted geometrically:

- Adding k *translates* the points on the line k units to the right (or $-k$ units to the left if $k < 0$);
- Multiplying by k *scales* the points on the line by a factor of $|k|$, and also *flips* it if $k < 0$.

But really, this 'flip' is rotation by π radians. What if we were allowed to rotate by arbitrary angles?



The plane swept out by the non-negative real line upon rotating it in a full circle is called the **complex plane**, and the points on it are called **complex numbers**.

Define a symbol i and interpret multiplication by i as rotation by $\frac{\pi}{2}$ radians. Then i^2 is rotation by π radians, which means that multiplication by i^2 and multiplication by -1 have the same effect. What this means (...take a deep breath...) is that

$$\boxed{i^2 = -1}$$

If we further interpret *addition* by i as translation *upwards* by 1 unit, then every complex number z can be obtained by *moving right* a units (for some real number a) and then *moving up* b units (for some real number b). Thus every complex number z has a unique representation as

$$z = a + bi$$

where a and b are real. The number a is called the **real part** of z, and the number b is called the **imaginary part** of z.

Arithmetic with complex numbers is then just like arithmetic with real numbers, except that now $i^2 = -1$. For example:

- $(a+bi)+(c+di) = (a+c)+(b+d)i$
- $(a+bi)(c+di) = ac+(ad+bc)i+bdi^2 = (ac-bd)+(ad+bc)i$

Fun facts about complex numbers:

- Every polynomial $f(x) = a_n x^n + \cdots + a_1 x + a_0$ with degree $\geqslant 1$ has n complex roots (counted with repeats). This is the *fundamental theorem of algebra*.
- If $f(x) = ax^2 + bx + c$ is a quadratic with real coefficients, then:
  - ◇ If $b^2 - 4ac > 0$, then f has two real roots;
  - ◇ If $b^2 - 4ac = 0$, then f has a repeated root;
  - ◇ If $b^2 - 4ac < 0$, then f has two complex roots, which are *complex conjugates* of one another—that is, their real parts are equal and the imaginary part of one root is the negative of that of the other.

  For example $x^2 - 4x + 8 = (x-2-2i)(x-2+2i)$. Its roots are $2+2i$ and $2-2i$, which are complex conjugates of one another.

Eigenvalues and eigenvectors

The upshot of this is that if we allow our eigenvalues to be *complex*, then the characteristic polynomial of **any** $n \times n$ matrix A can be completely factorised:

$$f_A(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$$

where $\lambda_1, \ldots, \lambda_n$ are complex numbers. Moreover, it is still true that

$$\lambda_1 \lambda_2 \cdots \lambda_n = \det(A) \quad \text{and} \quad \lambda_1 + \lambda_2 + \cdots + \lambda_n = \mathrm{tr}(A)$$

1. Perform the following tedious algebraic tasks.

(a) Find the roots of the polynomial $f(x) = x^2 + 4x + 8$.

(b) Find the square roots of the complex number $9i$.

(c) Let $z = a + bi$ be a complex number. Find $z + \bar{z}$ and $z\bar{z}$, where $\bar{z} = a - bi$ is the complex conjugate of z, and observe that $z + \bar{z}$ and $z\bar{z}$ are both real.

2. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

3. Show that if an 5×5 matrix A has characteristic polynomial

$$f_A(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + a_4\lambda^4 - \lambda^5$$

then $\det(A) = a_0$ and $\mathrm{tr}(A) = a_4$.

4. For each of the following statements, determine if it is always, sometimes or never true.

(a) Let A be a 2×2 real matrix. Then A can be diagonalised, provided the diagonal entries are allowed to be complex numbers.

(b) Let B be a 3×3 real matrix. Then B has exactly one non-real eigenvalue.

(c) Let C be an $n \times n$ real matrix and let λ be an eigenvalue of C. If every vector in E_λ is real, then λ is real.