Double integrals

A bounded integral $\int_{a}^{b} f(x) \, dx$ tells us the area under the curve $y = f(x)$ above the interval $[a, b] = \{x : a \leqslant x \leqslant b\}$. Intuitively, the integral adds up the heights of the points $(x, f(x))$ for $a \leqslant x \leqslant b$.

Double integrals are the generalisation of (bounded) integrals to functions of two variables: the double integral $\iint_{D} f(x, y) \, dA$ tells us the volume under the surface $z = f(x, y)$ above the region $D$ of the $(x, y)$-plane.

When $D$ is the square region $[a, b] \times [c, d] = \{(x, y) : a \leqslant x \leqslant b, \ c \leqslant y \leqslant d\}$ and $f$ is sufficiently well-behaved* on $D$, there are two ways that we can compute $\iint_{D} f(x, y) \, dA$:

- Find the areas under the curves $z = f(x, y)$ for fixed $a \leqslant x \leqslant b$ (by integrating with respect to $y$, holding $x$ constant); then ‘add up’ these areas by integrating with respect to $x$:

$$\int_{[a,b] \times [c,d]} f(x, y) \, dA = \int_{a}^{b} \left( \int_{c}^{d} f(x, y) \, dy \right) \, dx$$

- Find the areas under the curves $z = f(x, y)$ for fixed $c \leqslant y \leqslant d$ (by integrating with respect to $x$, holding $y$ constant); then ‘add up’ these areas by integrating with respect to $y$:

$$\int_{[a,b] \times [c,d]} f(x, y) \, dA = \int_{c}^{d} \left( \int_{a}^{b} f(x, y) \, dx \right) \, dy$$

Note that, in particular, the two iterated integrals are equal—this fact is called **Fubini’s theorem**.

[*Every function we will encounter is ‘sufficiently well-behaved’ for the purposes of applying Fubini’s theorem.*]
1. Compute \( \int_{[1,2] \times [-1,1]} xe^{xy} \, dA \ldots \)

(a) \ldots by first integrating with respect to \( y \) and then with respect to \( x \).

(b) \ldots by first integrating with respect to \( x \) and then with respect to \( y \).
2. Use double integration to show that the volume of a cube of width \( a \), length \( b \) and height \( c \) is equal to \( abc \).
3. Find the volume of the solid bounded by the \((x, y)\)-plane, the plane \(x = 1\), the plane \(x = -1\), the plane \(z = 1 + y\) and the plane \(z = 2 - y\).