Ortho-more-mal

Recall that vectors \( \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k \) in \( \mathbb{R}^n \) are orthonormal if and only if

\[
\vec{u}_i \cdot \vec{u}_j = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j
\end{cases}
\]

and that the **orthogonal projection** of a vector \( \vec{x} \) onto a subspace \( V \) with orthonormal basis \( \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k \) is given by

\[
\text{proj}_V(\vec{x}) = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2 + \cdots + (\vec{u}_k \cdot \vec{x})\vec{u}_k
\]

Consequently, if \( \mathcal{B} = \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n \) is an orthonormal basis of \( \mathbb{R}^n \), then \( \vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_n \cdot \vec{x})\vec{u}_n \).

This makes computing coordinate vectors with respect to orthonormal bases extremely easy:

\[
[x]_\mathcal{B} = \begin{pmatrix}
\vec{u}_1 \cdot \vec{x} \\
\vec{u}_2 \cdot \vec{x} \\
\vdots \\
\vec{u}_n \cdot \vec{x}
\end{pmatrix}
\]

The **orthogonal complement** of a subspace \( V \) of \( \mathbb{R}^n \) is the subspace \( V^\perp \) of \( \mathbb{R}^n \) consisting of all vectors perpendicular to those in \( V \):

\[
V^\perp = \{ \vec{x} \in \mathbb{R}^n : \vec{v} \cdot \vec{x} = 0 \text{ for all } \vec{v} \in V \} = \ker(\text{proj}_V)
\]

Note that \( \dim(V) + \dim(V^\perp) = n \) by the rank-nullity theorem, since \( V = \text{im}(\text{proj}_V) \).

Some geometry

Dot products, lengths and angles all neatly related by the following theorem: if \( \vec{x} \) and \( \vec{y} \) are any two vectors in \( \mathbb{R}^n \), such that the angle between \( \vec{x} \) and \( \vec{y} \) is \( \theta \) (where \( 0 \leq \theta \leq \pi \)), then

\[
\vec{x} \cdot \vec{y} = ||\vec{x}|| ||\vec{y}|| \cos \theta
\]

Some more fun facts:

- \( ||\vec{x}|| < ||\vec{x}|| ||\vec{y}|| \) — this is called the Cauchy–Schwarz inequality;
- \( ||\vec{x} + \vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2 \) if and only if \( \vec{x} \) and \( \vec{y} \) are orthogonal.
1. (a) Verify that \( \mathcal{B} = \begin{pmatrix} 1/2 \\ 0 \\ \sqrt{3}/2 \end{pmatrix}, \begin{pmatrix} -\sqrt{3}/2 \\ 0 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \) is an orthonormal basis of \( \mathbb{R}^3 \).

(b) Find the coordinates of \( \vec{a} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \) with respect to \( \mathcal{B} \).
2. Let $V$ be the plane in $\mathbb{R}^3$ spanned by \[
\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.
\]

(a) Find the orthogonal projection of \[
\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
\]
on $V$;

(b) Find the orthogonal complement of $V$. 
3. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be vectors in $\mathbb{R}^3$ defined by

$\vec{a} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

(a) Show that $\vec{c}$ is in the orthogonal complement of span\{$\vec{a}, \vec{b}$\}.

(b) Find the angle between $\vec{a}$ and $\vec{b}$. 
4. For each of the following (true) statements, explain why it is true.

(a) If $|\vec{x} \cdot \vec{y}| = \|\vec{x}\| \|\vec{y}\|$, then $\vec{x}$ and $\vec{y}$ are parallel.

(b) Let $\ell$ be a line in $\mathbb{R}^n$ and let $\vec{v}$ and $\vec{w}$ be nonzero vectors in $\mathbb{R}^n$. If $\vec{v}$ is parallel to $\ell$, and the equation $\|\vec{v} + \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2$ holds, then $\vec{w}$ is in $\ell^\perp$.

(c) Let $V$ be a subspace of $\mathbb{R}^n$ and let $\vec{x}$ be a vector in $\mathbb{R}^n$. Then $\|\text{proj}_V(\vec{x})\| \leq \|\vec{x}\|$.