Math 290-2 Class 3
Friday 11th January 2019

Gram–Schmidt orthonormalisation

We like orthonormal bases, but alas, not all bases are created orthonormal. The Gram–Schmidt process is a recursive procedure for turning a basis of (a subspace of) \( \mathbb{R}^n \) into an orthonormal basis (of the same subspace).

The procedure goes like this. Let \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \) be linearly independent vectors in \( \mathbb{R}^n \).

**Step 1.** Divide \( \vec{v}_1 \) by its length to obtain a unit vector \( \vec{u}_1 \) parallel to \( \vec{v}_1 \):

\[
\vec{u}_1 = \frac{1}{\| \vec{v}_1 \|} \vec{v}_1
\]

**Step 2.** Define \( \vec{v}_2^\perp = \vec{v}_2 - \text{proj}_{\vec{v}_1}(\vec{v}_2) \), and observe:

(a) \( \vec{v}_2^\perp \) is perpendicular to \( \vec{v}_1 \); and
(b) \( \vec{v}_2^\perp \) is in the span of \( \vec{v}_1 \) and \( \vec{v}_2 \);

And then divide \( \vec{v}_2^\perp \) by its length to obtain a unit vector \( \vec{u}_2 \) parallel to \( \vec{v}_2^\perp \):

\[
\vec{v}_2^\perp = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2) \vec{u}_1 \quad \text{and} \quad \vec{u}_2 = \frac{1}{\| \vec{v}_2^\perp \|} \vec{v}_2^\perp
\]

**Step 3.** Define \( \vec{v}_3^\perp = \vec{v}_3 - \text{proj}_{\vec{v}_1,\vec{v}_2}(\vec{v}_3) \), and observe:

(a) \( \vec{v}_3^\perp \) is perpendicular to \( \vec{v}_1 \) and \( \vec{v}_2 \); and
(b) \( \vec{v}_3^\perp \) is in \( \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \);

And then divide \( \vec{v}_3^\perp \) by its length to obtain a unit vector \( \vec{u}_3 \) parallel to \( \vec{v}_3^\perp \):

\[
\vec{v}_3^\perp = \vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_2) \vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3) \vec{u}_2 \quad \text{and} \quad \vec{u}_3 = \frac{1}{\| \vec{v}_3^\perp \|} \vec{v}_3^\perp
\]

...and so on... until:

**Step k.** Define \( \vec{v}_k^\perp = \vec{v}_k - \text{proj}_{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_{k-1}}(\vec{v}_k) \), and observe:

(a) \( \vec{v}_k^\perp \) is perpendicular to \( \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{k-1}\} \); and
(b) \( \vec{v}_k^\perp \) is in \( \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \);

And then divide \( \vec{v}_k^\perp \) by its length to obtain a unit vector \( \vec{u}_k \) parallel to \( \vec{v}_k^\perp \):

\[
\vec{v}_k^\perp = \vec{v}_k - (\vec{u}_1 \cdot \vec{v}_k) \vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_k) \vec{u}_2 - \cdots - (\vec{u}_{k-1} \cdot \vec{v}_k) \vec{u}_{k-1} \quad \text{and} \quad \vec{u}_k = \frac{1}{\| \vec{v}_k^\perp \|} \vec{v}_k^\perp
\]

**Result:** The vectors \( \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k \) are orthonormal, and \( \text{span}\{\vec{u}_1, \ldots, \vec{u}_j\} = \text{span}\{\vec{v}_1, \ldots, \vec{v}_j\} \) for all \( j \leq k \). Therefore, if \( \vec{v}_1, \ldots, \vec{v}_k \) is a basis of \( V \), then \( \vec{u}_1, \ldots, \vec{u}_k \) is an orthonormal basis of \( V \).
1. Use the Gram–Schmidt process to turn the following sets of vectors into orthonormal sets spanning the same subspace.

(a) \[
\begin{pmatrix}
1 \\
0 \\
-1
\end{pmatrix}
\]

(b) \[
\begin{pmatrix}
1 \\
0 \\
-1
\end{pmatrix}, \begin{pmatrix}
1 \\
1 \\
0
\end{pmatrix}
\]
(c) \[
\begin{pmatrix}
1 \\
0 \\
-1 \\
0
\end{pmatrix}, \begin{pmatrix}
1 \\
1 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
1 \\
0 \\
0 \\
1
\end{pmatrix}
\]
2. Find an orthonormal basis of the plane $x - 2y + z = 0$. 
3. Define a matrix $A$ by

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 2 & 3 & 1 & -1 \end{pmatrix}$$

(a) Find the rank of $A$.

(b) Find an orthonormal basis of the image of $A$.

(c) Find an orthonormal basis of the kernel of $A$. 