Math 290-2 Class 4
Monday 14th January 2019

Orthogonal transformations

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if $T$ preserves dot products—more precisely, if $T(\vec{x}) \cdot T(\vec{y}) = \vec{x} \cdot \vec{y}$ for all vectors $\vec{x}, \vec{y}$ in $\mathbb{R}^n$. Some fun facts about orthogonal transformations:

- $T$ is orthogonal $\iff$ $T$ preserves lengths—that is, $\|T(\vec{x})\| = \|\vec{x}\|$ for all vectors $\vec{x}$ in $\mathbb{R}^n$.
- If $T$ is orthogonal and $\vec{x}$ and $\vec{y}$ are perpendicular, then $T(\vec{x})$ and $T(\vec{y})$ are perpendicular.
- $T$ is orthogonal $\iff T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n)$ are an orthonormal basis of $\mathbb{R}^n$.

Transposes and orthogonal matrices

Recall that the transpose of an $m \times n$ matrix $A$ is the $n \times m$ matrix $A^T$, where the $(i, j)^{th}$ component of $A^T$ is the $(j, i)^{th}$ component of $A$:

$$
\begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix}^T = 
\begin{pmatrix}
1 & 3 \\
2 & 4
\end{pmatrix},
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{pmatrix}^T = 
\begin{pmatrix}
1 & 4 & 2 \\
2 & 5 & 0 \\
3 & 6 & 2
\end{pmatrix},
\begin{pmatrix}
1 & 2 & 3 \\
2 & 0 & 2 \\
3 & 2 & 1
\end{pmatrix}^T = 
\begin{pmatrix}
1 & 2 & 3 \\
2 & 0 & 2 \\
3 & 2 & 1
\end{pmatrix}
$$

Transposes satisfy some nice properties, like $(AB)^T = B^T A^T$ and $(A^T)^{-1} = (A^{-1})^T$. Suppose the columns of $A$ are $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Then

$$A^T A = \begin{pmatrix}
\cdots & v_1^T & \cdots \\
\cdots & v_2^T & \cdots \\
\vdots & \vdots & \vdots \\
\cdots & v_n^T & \cdots \\
\end{pmatrix} \begin{pmatrix}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{pmatrix} = \begin{pmatrix}
\vec{v}_1 \cdot \vec{v}_1 & \vec{v}_1 \cdot \vec{v}_2 & \cdots & \vec{v}_1 \cdot \vec{v}_n \\
\vec{v}_2 \cdot \vec{v}_1 & \vec{v}_2 \cdot \vec{v}_2 & \cdots & \vec{v}_2 \cdot \vec{v}_n \\
\vdots & \vdots & \ddots & \vdots \\
\vec{v}_n \cdot \vec{v}_1 & \vec{v}_n \cdot \vec{v}_2 & \cdots & \vec{v}_n \cdot \vec{v}_n
\end{pmatrix}$$

A matrix $A$ is orthogonal if the linear transformation $T(\vec{x}) = A\vec{x}$ is an orthogonal transformation. But $T$ is orthogonal if and only if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are orthonormal, since $\vec{v}_i = T(\vec{e}_i)$ for each $1 \leq i \leq n$. Therefore

$$A \text{ is orthogonal } \iff A^T A = I_n$$

since $\vec{v}_i \cdot \vec{v}_j = 1$ if $i = j$ and 0 if $i \neq j$. In particular, all orthogonal matrices are invertible.

Orthogonal projections... again

If $V$ has orthonormal basis $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$, then the matrix of orthogonal projection onto $V$ is given by $QQ^T$ (not $Q^T Q$!!), where $Q$ is the $n \times k$ matrix whose columns are $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$. 

Proofs

This section contains (relatively short) proofs of the results overleaf. You do not need to know these by heart, but you should be able to understand them.

- $T$ is orthogonal $\iff T$ preserves lengths.
  
  Proof. This is because the dot product can be expressed in terms of the length and vice versa:
  
  $$\|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}} \quad \text{and} \quad \vec{a} \cdot \vec{b} = \|\vec{a} + \vec{b}\|^2 - \|\vec{a}\|^2 - \|\vec{b}\|^2$$
  
  So if $T$ preserves one, then $T$ preserves the other.

- If $T$ is orthogonal, then $T$ preserves angles.
  
  Proof. If the angle between $\vec{x}$ and $\vec{y}$ is $\theta$, then
  
  $$\theta = \arccos\left(\frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|}\right) = \arccos\left(\frac{T(\vec{x}) \cdot T(\vec{y})}{\|T(\vec{x})\| \|T(\vec{y})\|}\right)$$
  
  so the angle between $T(\vec{x})$ and $T(\vec{y})$ is also $\theta$.

- If $T$ is orthogonal and $\vec{x}$ and $\vec{y}$ are perpendicular, then $T(\vec{x})$ and $T(\vec{y})$ are perpendicular.
  
  Proof. This follows from the fact that $T$ preserves angles, with $\theta = \frac{\pi}{2}$.

- If $T$ is orthogonal, then $T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n)$ are an orthonormal basis of $\mathbb{R}^n$.
  
  Proof.
  
  Since the vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ are orthonormal and $T$ preserves lengths and angles, the vectors $T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n)$ are also orthonormal.

- If $T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n)$ are an orthonormal basis of $\mathbb{R}^n$, then $T$ is orthogonal.
  
  Proof. Write $\vec{u}_i = T(\vec{e}_i)$ for each $1 \leq i \leq n$, so that $T(\vec{x}) = x_1 \vec{u}_1 + x_2 \vec{u}_2 + \cdots + x_n \vec{u}_n$ for each $\vec{x}$ in $\mathbb{R}^n$. Then
  
  $$T(\vec{x}) \cdot T(\vec{y}) = \left(\sum_{i=1}^n x_i \vec{u}_i\right) \cdot \left(\sum_{j=1}^n y_j \vec{u}_j\right) = \sum_{i=1}^n \sum_{j=1}^n x_iy_j (\vec{u}_i \cdot \vec{u}_j)$$
  
  by definition of dot product
  
  $$= \sum_{i=1}^n \sum_{j=1}^n x_iy_j (\vec{u}_i \cdot \vec{u}_j)$$
  
  by linearity
  
  $$= \sum_{i=1}^n x_iy_i$$
  
  since $\vec{u}_i \cdot \vec{u}_j = 0$ if $i \neq j$ and $1$ if $i = j$
  
  $$= \vec{x} \cdot \vec{y}$$
  
  by definition of dot product
  
  Hence $T$ is orthogonal.

- If $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$ is an orthonormal basis of a subspace $V$ of $\mathbb{R}^n$, then the matrix of $\text{proj}_V$ is $QQ^T$, where $Q$ is the $n \times k$ matrix with columns $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$.
  
  Proof. First notice that for all $1 \leq j \leq k$ we have $Q^T \vec{u}_j = \vec{e}_j$, and that if $\vec{w}$ is perpendicular to $V$ then $Q^T \vec{w} = \vec{0}$. Extend the orthonormal basis of $V$ to an orthonormal basis $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ of $\mathbb{R}^n$.
  
  If $j \leq k$ then $QQ^T \vec{u}_j = Q\vec{e}_j = \vec{u}_j = \text{proj}_V(\vec{u}_j)$, and if $j > k$ then $QQ^T \vec{u}_j = Q\vec{0} = \vec{0} = \text{proj}_V(\vec{u}_j)$.
  
  Since $QQ^T$ and $\text{proj}_V$ agree on a basis, we have $QQ^T(\vec{x}) = \text{proj}_V(\vec{x})$ for all $\vec{x}$ in $\mathbb{R}^n$. 

1. (a) Show that the matrix \[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta 
\end{pmatrix}
\] is orthogonal.

(b) Suppose that \(a^2 + b^2 = 1\). Show that the matrix \[
\begin{pmatrix}
a & b \\
b & -a 
\end{pmatrix}
\] is orthogonal.

(c) (Try at home:) Show that all orthogonal 2 \times 2 matrices are of the form (a) or (b). Hence all orthogonal transformations \(\mathbb{R}^2 \rightarrow \mathbb{R}^2\) are rotations and reflections.
2. For each of the following statements, determine whether it is always, sometimes or never true.

(a) Let $A$ be an orthogonal matrix. Then $\det(A) = \pm 1$.

(b) Let $A$ be a $2 \times 3$ matrix. Then $A^T A$ is orthogonal.

(c) Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation. If the angle between vectors $\vec{x}$ and $\vec{y}$ is $\theta$, then the angle between $T(\vec{x})$ and $T(\vec{y})$ is $\theta$. 

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(d) Suppose $\det(A) = 1$. Then $A$ is orthogonal.

(e) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

\[
T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

Then $T$ is orthogonal.
3. (a) Find an orthonormal basis of the plane $V$ in $\mathbb{R}^3$ described by the equation $2x + y - 3z = 0$.

(b) Find the matrix of orthogonal projection onto $V$. 