Coordinate systems (repeated from Monday)

The cartesian coordinates (named after René Descartes) of a point $P$ in $\mathbb{R}^n$ tell us how many units must be traversed from the origin in the direction of each standard basis vector to get to $P$. But this is not the only way of specifying a point—it may be more convenient to use a different coordinate system, such as one of those described below.

The polar coordinates of a point $P$ in $\mathbb{R}^2$ are given by $(r, \theta)$, where:

- $r$ is the distance of $P$ from the origin; and
- $\theta$ is the angle of the ray from the origin on which $P$ lies.

We may allow $r < 0$, in which case $P$ lies $|r|$ units on the ray with angle $\theta + \pi$.

The conversion between cartesian and polar coordinates is given by

$$
\begin{align*}
    x &= r \cos \theta \\
    y &= r \sin \theta \\
    r^2 &= x^2 + y^2 \\
    \tan \theta &= \frac{y}{x} \text{ or indeterminate if } x = 0
\end{align*}
$$

The cylindrical coordinates of a point $P$ in $\mathbb{R}^3$ are given by $(r, \theta, z)$, where:

- $(r, \theta)$ are the polar coordinates of the projection of $P$ onto the $(x,y)$-plane; and
- $z$ is the height of $P$ along the $z$-axis.

The conversion between cartesian and cylindrical coordinates is given by

$$
\begin{align*}
    x &= r \cos \theta \\
    y &= r \sin \theta \\
    z &= z \\
    r^2 &= x^2 + y^2 \\
    \tan \theta &= \frac{y}{x}
\end{align*}
$$

The spherical coordinates of a point $P$ in $\mathbb{R}^3$ are given by $(\rho, \varphi, \theta)$, where:

- $\rho$ is the distance of $P$ from the origin;
- $\varphi$ is the angle that $\overrightarrow{OP}$ makes with the positive $z$-axis; and
- $\theta$ is the angle of $P$ about the $z$-axis.

The conversion between cartesian and spherical coordinates is given by

$$
\begin{align*}
    x &= \rho \cos \theta \sin \varphi \\
    y &= \rho \sin \theta \sin \varphi \\
    z &= \rho \cos \varphi \\
    \rho^2 &= x^2 + y^2 + z^2 \\
    \tan \varphi &= \sqrt{\frac{x^2 + y^2}{z}} \\
    \tan \theta &= \frac{y}{x}
\end{align*}
$$
1. Let $C$ be the circle in the $(x,z)$ plane with centre $(2,0,0)$ and radius 1. Find the cylindrical coordinate equation of the torus traced by $C$ upon rotation by $2\pi$ radians about the $z$-axis.

2. Find the cartesian equation of the surface described in spherical coordinates by

$$\rho^2 (a \sin^2 \varphi \cos^2 \theta + b \sin^2 \varphi \sin^2 \theta + c \cos^2 \theta) = 1$$

Sketch this surface in when $a > b > c > 0$. 
3. Let $S$ be the cone whose cylindrical equation is $z = r$, $z \geq 0$. Describe $S$ in spherical coordinates.

4. Find a way of converting between cylindrical and spherical coordinates.
5. Sketch the solid region of $\mathbb{R}^3$ described in spherical coordinates by

$$1 \leq \rho^2 \leq 4, \quad 0 \leq \varphi \leq \frac{\pi}{6}, \quad 0 \leq \theta \leq 2\pi$$

and describe this region in cartesian and cylindrical coordinates.